Prof. Dr. V. Schmidt
C. Hirsch

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Stochastic networks

Problem set 6

Due date: December 6, 2011

Exercise 1

Let $d \geq 1$ and $a_1, a_2, \ldots, a_d \geq 1$ be positive integers. Write a computer program that first simulates (one realization of) Bernoulli bond percolation with activation probability p on the bounded cuboid $R = [0, a_1] \times \ldots \times [0, a_d] \cap \mathbb{Z}^d$ and then checks whether the output of this simulation contains an activated crossing from the hypersurface $x_1 = 0$ to the hypersurface $x_1 = a_1$. The event of occurrence of this kind of crossing will be denoted by H.

Use this program to obtain Monte-Carlo estimates for the probability $\mathbb{P}_p(H)$ for various values of $d, a_1, \ldots, a_d$. Specifically make empirical drawings of the function $p \mapsto \mathbb{P}_p(H)$ for $d \in \{2, 3\}$ and $a_1 = \ldots = a_d \in \{10, 20\}$. Give an interpretation of your results.

Hint. Use the coupling construction of bond percolation so that each simulation may be used for all p-values simultaneously.

Exercise 2

Consider Bernoulli bond percolation on $\mathbb{Z}^d$ with activation probability $p \in (0, 1)$ satisfying $\mathbb{P}_p(|C_o| \geq n) \leq 2\exp(-n^\alpha)$ for some $\alpha > 0$ and all $n \geq 1$. Let $a_1, a_2, \ldots, a_d \geq 1$, write $R_n = [0, a_1 n] \times \ldots, \times [0, a_d n] \cap \mathbb{Z}^d$ and prove that the probability of existence of an activated crossing inside R_n from the hypersurface $x_1 = 0$ to the hypersurface $x_1 = a_1 n$ tends to 0 as $n \to \infty$.

Exercise 3

Let G_1 be the two-dimensional quadratic lattice $\mathbb{Z}^2$ and let G_2 be the graph with vertex set $\mathbb{Z}^2$ but edges given by $\{\{z_1, z_2\} \in \mathbb{Z}^2 \times \mathbb{Z}^2 : |z_1 - z_2| \leq \sqrt{2}\}$. For $n > 0$ denote by $S_n = [0, n] \times [0, n] \cap \mathbb{Z}^2$ a discrete square. Consider Bernoulli *site* percolation on $\mathbb{Z}^2$ and denote by $H(S_n, G_i)$ the event of obtaining an activated horizontal crossing in S_n using the edges of G_i. Prove that $\mathbb{P}^s_{1/2}(H(S_n, G_1)) = 1 - \mathbb{P}^s_{1/2}(H(S_n, G_2))$.

Hint. Try the same approach as in bond percolation.

Exercise 4

Let $k, \ell \geq 1$ be positive integers and consider Bernoulli bond percolation on $\mathbb{Z}^2$ with activation probability $p = 1/2$. Let $A_{k,\ell}$ denote the event of occurrence of an activated crossing (i.e. from the inner boundary to the outer boundary) of the discrete annulus $\mathbb{Z}^2 \cap [-2^{k+\ell}, 2^{k+\ell}] \setminus [-2^k, 2^k]$. Prove that there exists a constant $0 < c < 1$ independent of k, ℓ such that $\mathbb{P}_{1/2}(A_{k,\ell}) \leq c^\ell$ for all $k, \ell \geq 1$.

Hint. Try the same approach as in the proof of $\theta(1/2) = 0$.