Prof. Dr. V. Schmidt
C. Hirsch

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Stochastic networks

Problem set 7

Due date: December 13, 2011

Exercise 1

Consider Bernoulli bond percolation on $\mathbb{Z}^2$ with activation probability $p = 1/2$. Prove that $h(m_1 + m_2 - 2n, 2n) \geq h(m_1, 2n)h(m_2, 2n)/2$ holds for all $n \geq 1$ and all $m_1, m_2 \geq 2n$.

Hint. Consider $m_1 \times 2n$ and $m_2 \times 2n$ rectangles intersecting in a $2n \times 2n$-square and use Harris's inequality (see Lemma 2.5).

Exercise 2

Consider Bernoulli bond percolation on $\mathbb{Z}^2$ with activation probability $p = 1/2$. Let $\partial(S_n) = ([-n, n]^2 \setminus [-(n-1), n-1]^2) \cap \mathbb{Z}^2$. Prove that there exists a constant $c > 0$ satisfying $\mathbb{P}_{1/2}(o \to \partial(S_{6n})) \geq c\mathbb{P}_{1/2}(o \to \partial(S_{2n}))$ for all $n \geq 1$

Hint. Exercise 1 yields a constant $c' > 0$ such that $h(12n, 2n) > c'$ for all $n \geq 1$.

Exercise 3

Let $n \geq 1$ and let $(\Omega_n, \mathcal{A}_n, \mathbb{P}_n)$ be the probability space defined in the lecture. Furthermore define $\mathbf{p}' = (p_1, p_2, \ldots, p_{n-1}, p'_n)$, where $p'_n \geq p_n$. For $A \in \mathcal{A}_n$ increasing prove that $\mathbb{P}_{\mathbf{p}'}(A) - \mathbb{P}_{\mathbf{p}}(A) = (p'_n - p_n)\beta_n(A)$ and use this relation to give an alternative proof of the Margulis-Russo formula, where $\beta_n(A) = \mathbb{P}_{\mathbf{p}}(\omega_n \text{ is pivotal for } A)$.

Hint. Use the coupling construction to compare $\mathbb{P}_{\mathbf{p}'}$ and $\mathbb{P}_{\mathbf{p}}$.

Exercise 4

Let $n \geq 1$ and let $(\Omega_n, \mathcal{A}_n, \mathbb{P}_n)$ be the probability space defined in the lecture and let $A \in \mathcal{A}_n$ be increasing. Prove the validity of the steps

$$\begin{aligned}\frac{d}{dp}\mathbb{P}_p(A) &= \frac{1}{p}\sum_{i=1}^{n} \mathbb{P}_p(\omega_i = 1 \text{ and } \omega_i \text{ is pivotal for } A)\\ &= \frac{1}{p}\mathbb{E}_p(N(A)|A)\mathbb{P}_p(A),\end{aligned}$$

where $N(A)$ is the (random) number of pivotal coordinates for A. Deduce that $\mathbb{P}_{p_2}(A) \leq (p_2/p_1)^n \mathbb{P}_{p_1}(A)$ for all $0 < p_1 \leq p_2 < 1$.