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# Stochastic networks

## Problem set 8

Due date: December 20, 2011

### Exercise 1

Consider Bernoulli bond percolation on $\mathbb{Z}^d$. Prove that the function $p \mapsto \theta(p)$ is continuous from the left on $(p_c, 1]$. You may use without proof that for $p > p_c$ the number of infinite activated clusters is 1 a.s.

*Hint.* Use the standard coupling construction. It may be useful to reduce the claim to the statement $\mathbb{P}(|C(p)| = \infty, |C(p')| < \infty \text{ for all } p' < p_c) = 0$.

### Exercise 2

Consider Bernoulli bond percolation on $\mathbb{Z}^2$ with activation probability $p \in (0,1)$. For $n \geq 1$ write $\pi(n) = \mathbb{P}(o \to \partial(S_n))$, where $\partial(S_n) = ([-n,n]^2 \setminus [-(n-1), n-1]^2) \cap \mathbb{Z}^2$. Prove that for all $x \in \mathbb{Z}^2$ such that $|x|$ is even (i.e. $|x| = 2n$ for some $n \geq 1$) it holds that $\mathbb{P}_p(o \to x) \leq \pi(|x|/2)^2$, where $|x| = \max(|x_1|, |x_2|)$.

*Hint.* Check that the event $\{o \to x\}$ implies the existence of an activated path starting from $o$ and leaving $S_n$ as well as the existence of an activated path starting from $x$ and leaving $x + S_n$.

### Exercise 3

Consider Bernoulli bond percolation on $\mathbb{Z}^2$ with activation probability $p = 1/2$. For $n \geq 1$ write $\pi(n) = \mathbb{P}(o \to \partial(S_n))$. Prove that there exists a constant $c_1 > 0$ such that for all $x \in \mathbb{Z}^2$ we have $\mathbb{P}(o \to x) \geq c_1 \pi(2|x|)\pi(|x|)$.

*Hint.* Convince yourself that there exists an activated path from $o$ to $x$ if each of the following events occurs

(a) there exists an activated path from $o$ that leaves $[-2|x|, 2|x|]^2$

(b) there exists an activated path from $x$ that leaves $x + [-|x|, |x|]^2$

(c) there exists an activated cycle inside $[-2|x|, 2|x|]^2$ surrounding $[-|x|, |x|]^2$

(d) there exist activated crossings in both vertical halves and both horizontal halves of $x + [-|x|, |x|]^2$.

### Exercise 4

Let $G = (\mathbb{Z}^2, E)$ be the square lattice and let $\{X_e\}_{e \in E}$ be a collection of iid random variables with $X_e \sim U([0,1])$. Define a sequence of subgraphs $\{G_i\}_{i \geq 0}$ of $G$ recursively as follows. Put $G_0 = (\{o\}, \emptyset)$ and if $G_i = (V_i, E_i)$ is already constructed define $G_{i+1} = (V_{i+1}, E_{i+1})$ by $V_{i+1} = V_i \cup \{x, y\}$ and $E_{i+1} = E_i \cup \{e_i\}$, where $e_i = \{x, y\}$ is chosen to be the edge $e$ with minimal $X_e$-value among all edges connecting $G_i$ to $\mathbb{Z}^2 \setminus G_i$. Prove that $\limsup_{i \geq 1} X_{e_i} = 1/2$ a.s.

*Hint.* First prove that in Bernoulli bond percolation with parameter $1/2$ there a.s. exist infinitely many $n$ such that $[-n, n]^2$ is surrounded by an activated cycle.