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WS 2011/2012
December 20, 2011

Stochastic networks

Problem set 9

Due date: January 10, 2012

Exercise 1

Let $G = (V, E)$ be a countable graph with maximal vertex degree $2 \leq \Delta < \infty$. Let $v_0 \in V$ be fixed. Then for $n, m, b \geq 1$ we write $\mathcal{A}_{nmb}$ for the set of connected subgraphs $A \subset G$ such that $v_0 \in A$, A is connected, A consists of precisely n vertices and m edges and such that there exist precisely b edges of G that have exactly one end point in A. Furthermore we write $a_{nmb} = \#\mathcal{A}_{nmb}$.

(a) Prove that if $a_{nmb} \neq 0$ then $\max(b, 2m) \leq \Delta n$.

(b) Consider Bernoulli bond percolation on G with activation probability $p \in (0, 1)$. Prove that $\mathbb{P}_p(|C_v| = n) = \sum_{m,b \geq 1} a_{nmb} p^m (1-p)^b$, where C_{v_0} denotes the activated cluster at v_0.

(c) Using parts (a) and (b) prove that $\sum_{m,b \geq 1} a_{nmb} \leq (p(1-p)^2)^{-n\Delta/2}$ holds for all $n \geq 1$.

(d) Prove that the number of subtrees of G that contain v_0 and consist of n vertices is bounded by $(27/4)^{n\Delta/2}$.

Exercise 2

Let $p \in (0, 1)$ and let $\{X_v\}_{v \in \mathbb{Z}^2}$ be a collection of iid $\{0, 1\}$-valued random variables with $\mathbb{P}(X_v = 1) = p$ for all $v \in \mathbb{Z}^2$. For each $v \in \mathbb{Z}^2$ define $Y_v = \min\{1, \sum_{\substack{w \in \mathbb{Z}^2 \\ |w-v| \leq 1}} X_w\}$.

(a) Prove that $\{Y_v\}_{v \in \mathbb{Z}^2}$ is 3-dependent.

(b) Compute $\mathbb{P}(Y_v = 1)$.

(c) Prove that the $\{Y_v\}_{v \in \mathbb{Z}^2}$ are *not* independent.

Exercise 3

Consider Bernoulli bond percolation on $\mathbb{Z}^2$ with activation probability $p \in (0, 1)$.

(a) Show $h_p(4n, n) \geq h_p(2n, n)^3 h_p(n, n)^2 \geq h_p(2n, n)^5$.

(b) Show $h_p(4n, 2n) \geq 1 - (1 - h_p(2n, n)^5)^2$.

(c) Let ξ be the positive root of $x = 1 - (1 - x^5)^2$. Prove that if there exist $n > 0$ with $h_p(2n, n) > \xi$, then there exist $\ell > 0$ with $h_p(2^{k+\ell+1}n, 2^{k+\ell}n) \geq 1 - 2^{-k}/50$ for all $k \geq 0$. Deduce $\theta(p) > 0$ under these assumptions.

Hint. For part (a) consider three $(2n) \times n$-rectangles intersecting in two $n \times n$ squares.

Wir wünschen allen Studenten ein besinnliches Weihnachtsfest und einen guten Rutsch in ein erfolgreiches Jahr 2012!