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WS 2012/2013

Stochastics II
Exercise Sheet 10

Due to: Wednesday, 9th of January 2013

Exercise 1 (4 Points)

Let $\{X(t),\ t \geq 0\}$ be a gamma process with parameters $b, p > 0$, i.e. $X(t) \sim \Gamma(b, pt)$. Show that $\{X(t),\ t \geq 0\}$ is a subordinator with Laplace exponent

$$\xi(u) = \int_0^\infty (1 - \exp(-uy))\nu(dy)$$

where $\nu(dy) = py^{-1}\exp(-by)dy$, $y > 0$. Note that the Laplace exponent of $\{X(t),\ t \geq 0\}$ is defined as the function $\xi : [0,\infty) \to [0,\infty)$ with $E\exp(-uX(t)) = \exp(-t\xi(u))$ for arbitrary $t, u \geq 0$.

Exercise 2 (4 Points)

Let $\{X(t),\ t \geq 0\}$ be a compound Poisson process with Lévy measure

$$\nu(dx) = \frac{\lambda\sqrt{2}}{\sigma\sqrt{\pi}}\exp(-\frac{x^2}{2\sigma^2})dx, \qquad x \in \mathbb{R}$$

where $\lambda, \sigma > 0$. Show that the process $\{\sigma W(N(t)),\ t \geq 0\}$ has the same finite dimensional distributions as X. Here $\{N(t),\ t \geq 0\}$ is a Poisson process with intensity 2λ and W a Wiener process independent of N.

Hint: $\int_{-\infty}^{\infty} \cos(sy)\exp(-\frac{y^2}{2a})dy = \sqrt{2\pi a}\exp(-\frac{as^2}{2})$, for $a > 0$ and $s \in \mathbb{R}$.

Exercise 3 (8 Points)

Let X be a random variable on some probability space $(\Omega, \mathcal{F}, P)$ and let $\mathcal{B} \subseteq \mathcal{F}$ be a σ-algebra. If $E|X| < \infty$ the conditional expectation of X w.r.t. $\mathcal{B}$ is defined as the $\mathcal{B}$-measurable random variable Y with the following property

$$\int_B Y(w)P(dw) = \int_B X(w)P(dw), \qquad \forall B \in \mathcal{B}.$$

Instead of Y we use the notation $E(X|\mathcal{B})$. **Such a random variable exists and is a.s. uniquely determined**. Show the following properties of $E(X|\mathcal{B})$:

(a) $E(X|\{\emptyset, \Omega\}) = EX$ and $E(X|\mathcal{F}) = X$ a.s.

(b) If $X \leq Z$ a.s. then $E(X|\mathcal{B}) \leq E(Z|\mathcal{B})$ a.s.

(c) If X is $\mathcal{B}$-measurable it holds $E(XZ|\mathcal{B}) = X \cdot E(Z|\mathcal{B})$ a.s.

(d) $E(c|\mathcal{B}) = c$ a.s. for every constant $c \in \mathbb{R}$.

(e) If $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \mathcal{F}$ it holds

$$E(E(X|\mathcal{B}_2)|\mathcal{B}_1) = E(E(X|\mathcal{B}_1)|\mathcal{B}_2) = E(X|\mathcal{B}_1) \quad a.s.$$

(f) If the σ-algebras $X^{-1}(\mathcal{B}_\mathbb{R})$ and $\mathcal{B}$ are independent then it holds $E(X|\mathcal{B}) = EX$ a.s.