Exercise 1 (7 Points)

Let $X$ be an ID random variable with Lévy characteristics $(a, b, \nu)$.

(a) Let $c : \mathbb{R} \to \mathbb{R}$ be an arbitrary function with the following asymptotical properties:

\[ c(x) = 1 + o(|x|), \quad |x| \to 0 \]
\[ c(x) = O\left(\frac{1}{|x|}\right), \quad |x| \to \infty \]

Show that the characteristic function $\varphi$ of $X$ can be written in the form

\[ \varphi(z) = \exp\left(\frac{1}{2}\tilde{a}z^2 + i\tilde{b}c(z) + \int_{\mathbb{R}} \left(e^{ixz} - 1 - i x c(x)\right) \nu(dx)\right) \]

where $\tilde{a} \geq 0$ and $\tilde{b} \in \mathbb{R}$.

(b) Show that the following functions $\rho, \sigma, \tau : \mathbb{R} \to \mathbb{R}$ fulfill the conditions (1) and (2):

\[ \rho(x) = 1/(1 + x^2) \]
\[ \sigma(x) = \sin(x)/x \]
\[ \tau(x) = \begin{cases} 1 & : |x| \leq 1 \\ 1/|x| & : |x| > 1 \end{cases} \]

(c) Find three more examples (different from $\rho, \sigma, \tau$) for a function $c : \mathbb{R} \to \mathbb{R}$ which fulfill the conditions (1) and (2).

Exercise 2 (6 Points)

(a) Let $P(s) = \sum_{k=0}^{\infty} p_k s^k$ where $p_k \geq 0$ and $\sum_{k=0}^{\infty} p_k = 1$. Assume $P(0) > 0$ and that $\log(P(s))$ is a power series with positive coefficients. If $\varphi$ is the characteristic function of an arbitrary distribution $F$ show that $P(\varphi)$ is an ID characteristic function.

**Hint:** Find the Lévy measure in terms of $F^{**}$.

(b) Let $\varphi$ be a characteristic function. Show that $\psi : \mathbb{R} \to \mathbb{C}$ defined by

\[ \psi(z) = \frac{1 - b}{1 - a} \frac{1 - a\varphi(z)}{1 - b\varphi(z)}\]

$0 \leq a < b < 1$, is an ID characteristic function.

Exercise 3 (3 Points)

Show that the subordinator $T$ with marginal density

\[ f_T(t)(x) = t^{-\frac{3}{2}} e^{-t/4} I\{x > 0\} \]

is a $\frac{1}{2}$-stable subordinator.

**Hint:** Differentiate the Laplace transform of $T(t)$ and solve the differential equation.