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WS 2012/2013

Stochastics II
Exercise Sheet 11

Due to: Wednesday, 16th of January 2013

Exercise 1 (7 Points)

Let X be an ID random variable with Lévy characteristics (a, b, ν).

(a) Let $c : \mathbb{R} \to \mathbb{R}$ be an arbitrary function with the following asymptotical properties:

$$c(x) = 1 + o(|x|), \quad |x| \to 0 \tag{1}$$

$$c(x) = \mathcal{O}(1/|x|), \quad |x| \to \infty \tag{2}$$

Show that the characteristic function φ of X can be written in the form

$$\varphi(z) = \exp\left[-\frac{1}{2}\tilde{a}z^2 + iz\tilde{b}_c + \int_{\mathbb{R}} \left(e^{izx} - 1 - izxc(x)\right)\nu(dx)\right]$$

where $\tilde{a} \geq 0$ and $\tilde{b}_c \in \mathbb{R}$.

(b) Show that the following functions $\rho, \sigma, \tau : \mathbb{R} \to \mathbb{R}$ fullfill the conditions (1) and (2):

$$\rho(x) = 1/(1 + x^2)$$
$$\sigma(x) = \sin(x)/x$$
$$\tau(x) = \begin{cases} 1 & ; \ |x| \leq 1 \\ 1/|x| & ; \ |x| > 1 \end{cases}$$

(c) Find three more examples (different from ρ, σ, τ) for a function $c : \mathbb{R} \to \mathbb{R}$ which fullfill the conditions (1) and (2).

Exercise 2 (6 Points)

(a) Let $P(s) = \sum_{k=0}^{\infty} p_k s^k$ where $p_k \geq 0$ and $\sum_{k=0}^{\infty} p_k = 1$. Assume $P(0) > 0$ and that $\log(\frac{P(s)}{P(0)})$ is a power series with positive coefficients. If φ is the characteristic function of an arbitrary distribution F show that $P(\varphi)$ is an ID characteristic function.
Hint: Find the Lévy measure in terms of F^{*n}.

(b) Let φ be a characteristic function. Show that $\psi : \mathbb{R} \to \mathbb{C}$ defined by

$$\psi(z) = \frac{1-b}{1-a} \cdot \frac{1 - a\varphi(z)}{1 - b\varphi(z)},$$

$0 \leq a < b < 1$, is an ID characteristic function.

Exercise 3 (3 Points)

Show that the subordinator T with marginal density

$$f_{T(t)}(x) = \frac{t}{2\sqrt{\pi}} x^{-\frac{3}{2}} e^{-\frac{t^2}{4x}} \mathbb{1}\{x > 0\}$$

is a $\frac{1}{2}$-stable subordinator.
Hint: Differentiate the Laplace transform of $T(t)$ and solve the differential equation.