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WS 2012/2013

Stochastics II
Exercise Sheet 12

Due to: Wednesday, 23rd of January 2013

Exercise 1 (2 Points)

Let $\xi_1, \ldots, \xi_n$ be a sequence of i.i.d. random variables and $S_n = \sum_{k=1}^n \xi_k$. Calculate $E(\xi_n|S_n)$ $[= E(\xi_n|\sigma(S_n))]$.

Exercise 2 (3 Points)

We flip a coin n times and notice $Y_n = \mathbb{1}_{\{\text{the n-th flip is pitch}\}}$. Define

$$S_n = \sum_{k=1}^n Y_k.$$

Show that $\{X_n,\ n \in \mathbb{N}\}$ given by $X_n = 2S_n - n$ is a martingale w.r.t. the natural filtration.

Exercise 3 (4 Points)

Let $\{X_n,\ n \in \mathbb{N}\}$ be (independent) coin flippings, i.e. $P(X_n = 1) = p \in [0,1]$, $P(X_n = -1) = 1 - p$. Let $a > 0$ the seed capital and e_1 the money that we set before the first flip. For $n \geq 2$ define $e_n = C_{n-1}(X_1, \ldots, X_{n-1})$ with some function $C_{n-1} : \{-1,1\}^{n-1} \to \mathbb{R}_+$. Our financial siuation after the n-th flip is modeled by the random variables $S_n = S_{n-1} + X_n \cdot C_{n-1}(X_1, \ldots, X_{n-1}),\ n \geq 2$ and $S_1 = a + X_1 \cdot e_1$. Show that $\{S_n,\ n \in \mathbb{N}\}$ is a

$$\begin{cases} \text{supermartingale} & \text{if } p < \frac{1}{2} \\ \text{martingale} & \text{if } p = \frac{1}{2} \\ \text{submartingale} & \text{if } p > \frac{1}{2} \end{cases}$$

w.r.t. the natural filtration $\{\sigma(X_1, \ldots, X_n)\}_{n \in \mathbb{N}}$.

Exercise 4 (3 Points)

Let X and Y be arbitrary random variables on some probability space $(\Omega, \mathcal{F}, P)$. Define $E(Y|X) := E(Y|\sigma(X))$, where $\sigma(X) := \sigma(\{X^{-1}(B), B \in \mathcal{B}(\mathbb{R})\})$. The conditional variance of Y given X is defined by

$$Var(Y|X) := E((Y - E(Y|X))^2|X)$$

Show that

$$Var(Y) = E(Var(Y|X)) + Var(E(Y|X)).$$

Hint: The conditional expectation is linear!

Exercise 5 (6 Points)

For a stopping time τ we define the stopped σ-algebra $\mathcal{F}_\tau$ by

$$\mathcal{F}_\tau := \{B \in \mathcal{F} :\ B \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for arbitrary } t \geq 0\}.$$

Let ρ and γ be stopping times w.r.t. the filtration $\{\mathcal{F}_t,\ t \geq 0\}$. Show the following statements:

(a) $A \cap \{\rho \leq \gamma\} \in \mathcal{F}_\gamma, \forall\ A \in \mathcal{F}_\rho$.

(b) $\mathcal{F}_{\min\{\rho,\gamma\}} = \mathcal{F}_\rho \cap \mathcal{F}_\gamma$.