Exercise 1 (5 Points)
Let \( X = \{X(t), t \geq 0\} \) be càdlàg and adapted. Show that

\[
P\left( \sup_{0 \leq s \leq t} |X(s)| > x \right) \leq \frac{E(X(t))^2}{x^2 + E(X(t))^2}
\]

for arbitrary \( x > 0 \) and \( t \geq 0 \) if \( X \) is a submartingale with \( E(X(t)) = 0 \) and \( E(X(t))^2 < \infty \), \( \forall t \geq 0 \).

Exercise 2 (7 Points)

(a) Let \( g : [0, \infty) \to [0, \infty) \) a monotonously increasing function with

\[
\frac{g(x)}{x} \to \infty, \ (x \to \infty).
\]

Show that the sequence \( \{X_n\}_{n \in \mathbb{N}} \) of random variables is uniformly integrable if \( \sup_{n \in \mathbb{N}} E(g(|X_n|)) < \infty \).

(b) Let \( X = \{X_n\}_{n \in \mathbb{N}} \) be a martingale. Show that \( \{X_{T \wedge n}\}_{n \in \mathbb{N}} \) is uniformly integrable for every finite stopping time \( T : \Omega \to \mathbb{N} \), if \( E|X_T| < \infty \) and \( E(|X_n| \mathbb{1}_{\{T>n\}}) \to 0 \) for \( n \to \infty \).

Exercise 3 (4 Points)
Let \( S = \{S_n := a + \sum_{i=1}^{n} X_i, \ n \in \mathbb{N}\} \) be a symmetric random walk with \( a > 0 \) and \( P(X_i = 1) = P(X_i = -1) = \frac{1}{2}, \ i \in \mathbb{N} \). Show that \( \{M_n = \sum_{i=0}^{n} S_i - \frac{1}{2} S_n^2\}_{n \in \mathbb{N}} \) is a martingale (w.r.t. the natural filtration).

**Hint:** It suffices to show that \( E(M_{n+1}|\mathcal{F}_n) = M_n \), for all \( n \in \mathbb{N} \).