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Stochastics II
Exercise Sheet 13

Due to: Wednesday, 30th of January 2013

Exercise 1 (5 Points)

Let $X = \{X(t),\ t \geq 0\}$ be càdlàg and adapted. Show that

$$P(\sup_{0\leq s\leq t} |X(s)| > x) \leq \frac{E(X(t))^2}{x^2 + E(X(t))^2}$$

for arbitrary $x > 0$ and $t \geq 0$ if X is a submartingale with $E(X(t)) = 0$ and $E(X(t))^2 < \infty$, $\forall\ t \geq 0$.

Exercise 2 (7 Points)

(a) Let $g : [0, \infty) \to [0, \infty)$ a monotonously increasing function with

$$\frac{g(x)}{x} \to \infty,\ (x \to \infty).$$

Show that the sequence $\{X_n\}_{n\in\mathbb{N}}$ of random variables is uniformly integrable if $\sup\limits_{n\in\mathbb{N}} E(g(|X_n|)) < \infty$.

(b) Let $X = \{X_n\}_{n\in\mathbb{N}}$ be a martingale. Show that $\{X_{T\wedge n}\}_{n\in\mathbb{N}}$ is uniformly integrable for every finite stopping time $T : \Omega \to \mathbb{N}$, if $E|X_T| < \infty$ and $E(|X_n|\mathbb{1}_{\{T>n\}}) \to 0$ for $n \to \infty$.

Exercise 3 (4 Points)

Let $S = \{S_n := a + \sum\limits_{i=1}^{n} X_i,\ n \in \mathbb{N}\}$ be a symmetric random walk with $a > 0$ and $P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$, $i \in \mathbb{N}$. Show that $\{M_n = \sum\limits_{i=0}^{n} S_i - \frac{1}{3}S_n^3\}_{n\in\mathbb{N}}$ is a martingale (w.r.t. the natural filtration).
Hint: It suffices to show that $E(M_{n+1}|\mathcal{F}_n) = M_n$, for all $n \in \mathbb{N}$.