ulm university universität uulm

Prof. Dr. Evgeny Spodarev
Dipl.-Math. Stefan Roth

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Stochastics II
Exercise Sheet 14

Due to: Wednesday, 6th of February 2013

Exercise 1 (4 Points)

Let $\{X_n\}_{n\in\mathbb{N}}$ be a discrete martingale and T be a discrete stopping time w.r.t. $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$. Show that $\{X_{T\wedge n}\}_{n\in\mathbb{N}}$ is a martingale w.r.t. $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$.

Exercise 2 (7 Points)

Let $X_1, X_2, \ldots$ be i.i.d. with $P(X_1 = 1) = P(X_1 = -1) = 1/2$ and

$$S_n = \sum_{k=1}^{n} X_k, \quad n \in \mathbb{N}.$$

Define $T = \inf\{n : \ |S_n| > \sqrt{n}\}$ and $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, $n \in \mathbb{N}$.

(a) Show that T is a stopping time w.r.t. $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$.

(b) Show that $\{G_n\}_{n\in\mathbb{N}}$ with $G_n = S^2_{T\wedge n} - T \wedge n$ is a martingale w.r.t. $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$.
Hint: See Exercise 1.

(c) Show that $|G_n| \leq 4T$ for all $n \in \mathbb{N}$.
Hint: It holds $|G_n| \leq |S^2_{T\wedge n}| + |T \wedge n| \leq S^2_{T\wedge n} + T$.

Exercise 3 (5 Points)

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with $E|X_1| < \infty$ and $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, $n \in \mathbb{N}$. Furthermore let T be a stopping time w.r.t. $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ with $E(T) < \infty$.

(a) Let T be independent of $X_1, X_2, \ldots$. Give a formula for the characteristic function of $S_T = \sum_{k=1}^{T} X_k$. Use this formula to prove the identity of Wald, i.e. $E(S_T) = E(T)E(X_1)$.

(b) Now set $E(X_1) = 0$ and $T = \inf\{n : \ S_n < 0\}$. Show that $E(T) = \infty$.
Hint: Assume that $E(T) < \infty$ and find a contradiction.

Exercise 4 (8 Points, additional exercise)

One can show that for an arbitrary r.v. Y and an integrable r.v. X on a common probability space $(\Omega, \mathcal{F}, P)$ there exists a measurable function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $E(X|Y) \overset{a.s.}{=} \varphi(Y)$. We set $E(X|Y = y) := \varphi(y)$.

(a) Now let Y be discrete with values in $\{y_n, \ n \in \mathbb{N}\}$. Show that

$$\varphi(y) = \begin{cases} \int\limits_{\{\omega:\ Y(\omega)=y_k\}} X(\omega) \frac{P(d\omega)}{P(Y=y_k)} & \text{, if } y = y_k \text{ for some } k \\ 0 & \text{, otherwise.} \end{cases}$$

(b) Let X and Y be absolutely continuous with common density $f_{X,Y}$. Show that

$$\varphi(y) = \begin{cases} \int\limits_{\mathbb{R}} t \frac{f_{X,Y}(t,y)}{f_Y(y)} dt & \text{, } \forall\ y \text{ with } f_Y(y) > 0 \\ 0 & \text{, otherwise.} \end{cases}$$

Hint (a) and (b): It suffices to show that $\int\limits_{Y^{-1}(B)} X(\omega)P(d\omega) = \int\limits_{B} \varphi(y)P_Y(dy)$, $\forall\ B \in \mathcal{B}(\mathbb{R})$

(c) Let X and Y be independent and $X \sim Poi(\lambda)$, $Y \sim Poi(\mu)$, $\lambda, \mu > 0$. Calculate $E(X|Z = z)$, with $Z = X + Y$.