



## Stochastics II Exercise Sheet 15

Due to: Wednesday, 13th of February 2013

### Exercise 1 (5 Points)

Let  $\{X_n\}_{n \in \mathbb{N}}$  be a sequence of random variables, such that  $S_n := \sum_{k=1}^n X_k$  converges almost surely for  $n \rightarrow \infty$ . Furthermore let  $\{a_n\}_{n \in \mathbb{N}}$  be a monotonously increasing sequence of non-negative real numbers with  $a_n \xrightarrow{n \rightarrow \infty} \infty$ . Show that

$$\frac{1}{a_n} \sum_{k=1}^n a_k X_k \xrightarrow{n \rightarrow \infty} 0, \quad a.s.$$

**Hint:** It holds  $\frac{1}{a_n} \sum_{k=1}^n a_k X_k = \frac{1}{a_n} \sum_{k=1}^n (a_k - a_{k-1})(S_n - S_{k-1})$ ,  $a_0 := 0$ ,  $S_0 := 0$ .

### Exercise 2 (3 Points)

Let  $X$  be a non-negative random variable on some probability space  $(\Omega, \mathcal{F}, P)$  and  $T : \Omega \rightarrow \Omega$  a measure preserving map. Show that

$$\sum_{k=1}^{\infty} X(T^k(\omega)) = \infty,$$

for almost all  $\omega \in \Omega$  with  $X(\omega) > 0$ .

**Hint:** Theorem 6.2.1 could be helpful.

### Exercise 3 (4 Points)

Let  $X$  be a non-negative random variable on some probability space  $(\Omega, \mathcal{F}, P)$  and  $T : \Omega \rightarrow \Omega$  a measure preserving map. Show that  $E(X) = E(X \circ T)$ , i.e.

$$\int_{\Omega} X(T(\omega)) P(d\omega) = \int_{\Omega} X(\omega) P(d\omega).$$

**Hint:** Algebraic induction!

### Exercise 4 (4 Points)

Let  $(\Omega, \mathcal{F}, P) = ([0, 1), \mathcal{B}([0, 1)), \nu)$ , where  $\nu$  denotes the Lebesgue measure on  $[0, 1)$ . Let  $\lambda \in (0, 1)$ .

(a) Show that  $T(x) = (x + \lambda) \pmod{1}$  is a measure preserving map, where  $a \pmod{b} := a - \lfloor \frac{a}{b} \rfloor \cdot b$  for  $a \in \mathbb{R}$  and  $b \in \mathbb{Z}$ .

(b) Show that  $T(x) = \lambda x$  and  $T(x) = x^2$  are not measure preserving.