Exercise 1 (2 Points)
Show Remark 2.1.4: For a delayed renewal-process \( N, t \geq 0 \) with delay \( T_i \) it holds
(a) \( H(t) = \sum_{n=0}^{\infty} \left( F_{T_1} * F_{T_2} * \cdots * F_{T_n} \right)(t) \), \( t \geq 0 \).
(b) \( \tilde{i}_0(s) = \frac{i_0(s)}{\tilde{i}_0(0)} \), \( s > 0 \).

Exercise 2 (4 Points)
Let \( N = \{ N(t), t \geq 0 \} \) be a Poisson-process with intensity \( \lambda > 0 \), i.e. the interarrival times \( \{ T_n \}_{n \in \mathbb{N}} \) are exponentially distributed with mean \( \lambda^{-1} \). Calculate
\[
P(N(s) = n | N(t) = n)
\]
for \( s < t \) and \( n = 1, \ldots, n \).

**Hint:** You can use that the Poisson-process has independent increments without proof.

Exercise 3 (4 Points)
Let \( N^{(1)} = \{ N^{(1)}(t), t \geq 0 \} \) and \( N^{(2)} = \{ N^{(2)}(t), t \geq 0 \} \) be two independent Poisson-processes with intensities \( \lambda_1, \lambda_2 > 0 \), i.e. the sequences \( T_1^{(1)}, T_2^{(1)}, \ldots \) and \( T_1^{(2)}, T_2^{(2)}, \ldots \) are independent. Show that \( N = \{ N(t), t \geq 0 \} \) defined by
\[
N(t) = N^{(1)}(t) + N^{(2)}(t), \quad t \geq 0
\]
is a Poisson-process with intensity \( \lambda_1 + \lambda_2 \).

**Hint:** You can use that the Poisson-process has stationary increments without proof.

Exercise 4 (6 Points)
**Excursion:** Let \( \mathcal{B}_d(\mathbb{R}^d) := \{ B \in \mathcal{B}(\mathbb{R}^d), \nu_d(B) < \infty \} \), where \( \nu_d \) denotes the \( d \)-dimensional Lebesgue-measure. The non-homogeneous Poisson-process can be defined as \( N = \{ N(B), B \in \mathcal{B}(\mathbb{R}^d) \} \) with the following properties:
1.) \( N(B) \sim \text{Poi}(\mu(B)) \) for every \( B \in \mathcal{B}_d(\mathbb{R}^d) \).
2.) \( N(B_1), \ldots, N(B_n) \) are independent random variables for pairwise disjoint \( B_i \in \mathcal{B}_d(\mathbb{R}^d) \), \( i = 1, \ldots, n \) and arbitrary \( n \in \mathbb{N} \).

for a locally finite measure \( \mu : \mathcal{B}(\mathbb{R}^d) \to [0, \infty] \) (i.e. \( \mu(B) < \infty \) for every \( B \in \mathcal{B}(\mathbb{R}^d) \)), \( \mu \) is called the intensity measure of \( N \).
Let \( n \in \mathbb{N}, k_1, \ldots, k_n \in \mathbb{N}_0 \) and \( B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R}^d) \) pairwise disjoint.

(a) Show that
\[
P(N(B_1) = k_1, \ldots, N(B_n) = k_n) = \frac{\mu^{k_1}(B_1) \cdots \mu^{k_n}(B_n)}{k_1! \cdots k_n!} \exp\left( -\sum_{i=1}^{n} \mu(B_i) \right)
\]
(b) Verify that for \( k = \sum_{i=1}^{n} k_i \) and \( B = \bigcup_{i=1}^{n} B_i \) it holds
\[
P(N(B) = k) = \frac{k!}{k_1! \cdots k_n!} \mu^{k_1}(B_1) \cdots \mu^{k_n}(B_n)
\]
provided that \( \mu(B) > 0 \).