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WS 2012/2013

Stochastics II
Exercise Sheet 5

Due to: Wednesday, 21st of November 2012

Exercise 1 (2 Points)

Show Remark 2.1.4: For a delayed renewal-process $N = \{N(t),\ t \geq 0\}$ with delay T_1 it holds

(a) $H(t) = \sum_{n=0}^{\infty} (F_{T_1} * F_{T_2}^{*n})(t),\ \ t \geq 0.$

(b) $\hat{l}_H(s) = \frac{\hat{l}_{T_1}(s)}{1-\hat{l}_{T_2}(s)},\ \ s > 0$

Exercise 2 (4 Points)

Let $N = \{N(t),\ t \geq 0\}$ be a Poisson-process with intensity $\lambda > 0$, i.e. the interarrival times $\{T_n\}_{n\in\mathbb{N}}$ are exponentially distributed with mean λ^{-1}. Calculate

$$P(N(s) = i | N(t) = n)$$

for $s < t$ and $i = 1, \dots, n$.
Hint: You can use that the Poisson-process has independent increments without proof.

Exercise 3 (4 Points)

Let $N^{(1)} = \{N^{(1)}(t),\ t \geq 0\}$ and $N^{(2)} = \{N^{(2)}(t),\ t \geq 0\}$ be two independent Poisson-processes with intensities $\lambda_1, \lambda_2 > 0$, i.e. the sequences $T_1^{(1)}, T_2^{(1)}, \dots$ and $T_1^{(2)}, T_2^{(2)}, \dots$ are independent. Show that $N = \{N(t),\ t \geq 0\}$ defined by

$$N(t) = N^{(1)}(t) + N^{(2)}(t), \qquad t \geq 0$$

is a Poisson-process with intensity $\lambda_1 + \lambda_2$.
Hint: You can use that the Poisson-process has stationary increments without proof.

Exercise 4 (6 Points)

Excursion: Let $\mathcal{B}_0(\mathbb{R}^d) := \{B \in \mathcal{B}(\mathbb{R}^d),\ \nu_d(B) < \infty\}$, where ν_d denotes the d-dimensional Lebesgue-measure. The non-homogeneous Poisson-process can be defined as $N = \{N(B),\ B \in \mathcal{B}(\mathbb{R}^d)\}$ with the following properties:

1.) $N(B) \sim Poi(\mu(B))$ for every $B \in \mathcal{B}_0(\mathbb{R}^d)$.

2.) $N(B_1), \dots, N(B_n)$ are independent random variables for pairwise disjoint $B_i \in \mathcal{B}_0(\mathbb{R}^d)$, $i = 1, \dots, n$ and arbitrary $n \in \mathbb{N}$.

for a locally finite measure $\mu : \mathcal{B}(\mathbb{R}^d) \to [0, \infty]$ (i.e. $\mu(B) < \infty$ for every $B \in \mathcal{B}_0(\mathbb{R}^d)$). μ is called the intensity measure of N.
Let $n \in \mathbb{N}$, $k_1, \dots, k_n \in \mathbb{N}_0$ and $B_1, ..., B_n \in \mathcal{B}_0(\mathbb{R}^d)$ pairwise disjoint.

(a) Show that

$$P(N(B_1) = k_1, \dots, N(B_n) = k_n) = \frac{\mu^{k_1}(B_1) \dots \mu^{k_n}(B_n)}{k_1! \dots k_n!} \exp(-\sum_{i=1}^{n} \mu(B_i))$$

(b) Verify that for $k = \sum_{i=1}^{n} k_i$ and $B = \bigcup_{i=1}^{n} B_i$ it holds

$$P(N(B_1) = k_1, \dots, N(B_n) = k_n | N(B) = k) = \frac{k!}{k_1! \dots k_n!} \frac{\mu^{k_1}(B_1) \dots \mu^{k_n}(B_n)}{\mu^k(B)}$$

provided that $\mu(B) > 0$.