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# Stochastics II
# Exercise Sheet 6

Due to: Wednesday, 28th of November 2012

**Exercise 1** (7 Points)

Let $N = \{N(t),\ t \geq 0\}$ be a renewal process. Define $T(t) = S_{N(t)+1} - t$, $C(t) = t - S_{N(t)}$, $t > 0$ and $D(t) = T(t) + C(t)$, $t > 0$. $T$ is called **excess time**, $C$ **current life time** and $D$ **life time**. Now let $N$ be a Poisson process with intesity $\lambda > 0$.

(a) Calculate the distribution of $T(t)$.

(b) Show, that the distribution of the current life time is given by

$$P(C(t) \leq s) = \exp(-\lambda t)\delta_t(s) + \int_0^s f_{C(t)|N(t)>0}(x)dx, \quad s \in [0, t]$$

with $f_{C(t)|N(t)>0}(x) = \lambda \exp(-\lambda x)\mathbb{1}\{x \leq t\}$.

(c) Show that $P(D(t) \leq x) = (1 - (1 + \lambda \min\{t, x\}) \exp(-\lambda x))\mathbb{1}\{x \geq 0\}$.

(d) To determine $ET(t)$ consider the following argumenation: On average $t$ lies half between $S_{N(t)}$ and $S_{N(t)+1}$, i.e. $ET(t) = \frac{1}{2}E(S_{N(t)+1} - S_{N(t)}) = \frac{1}{2}ET_{N(t)+1} = \frac{1}{2\lambda}$. From part (a) we know that this is not true. Why?

**Exercise 2** (4 Points)

Let $X = \{X(t),\ t \geq 0\}$ be a compound Poisson process,

$$X(t) = \sum_{i=1}^{N(t)} U_i\ , \qquad t \geq 0$$

where $\{U_n\}_{n\in\mathbb{N}}$ is a sequence of non-negative i.i.d. random variables which are independent of $N$, and $N$ is a homogeneous Poisson process. Show that the Laplace transform of $X(t)$ is given by

$$\hat{l}_{X(t)}(s) = m_{N(t)}(\hat{l}_{U_1}(s)), \qquad s \geq 0$$

where $m_{N(t)} = Es^{N(t)}$ denotes the generating function of $N(t)$.

**Exercise 3** (3 Points)

Let $X = \{X(t),\ t \geq 0\}$ be a compound Poisson process like in Exercise 2, with $U_1 \sim Exp(\gamma)$. Show, that

$$\hat{l}_{X(t)}(s) = \exp(-\frac{\lambda ts}{\gamma + s}), \qquad s \geq 0$$

where $\lambda > 0$ denotes the intensity of $N(t)$.

**Exercise 4** (2 Points)

Consider a Poisson counting measure $N$ as in Exercise 4, Sheet 5. Let $B_0 \in \mathcal{B}(\mathbb{R}^d)$ and $\tilde{\mu}(\cdot) := \mu(\cdot \cap B_0)$. Show, that the random counting measure $\tilde{N} = \{\tilde{N}(B),\ B \in \mathcal{B}(\mathbb{R}^d)\}$ defined by $\tilde{N}(\cdot) := N(\cdot \cap B_0)$ is a Poisson counting measure with intensity measure $\tilde{\mu}$.