Exercise 1 (7 Points)
Let $N = \{ N(t), t \geq 0 \}$ be a renewal process. Define $T(t) = S_{N(t)+1} - t$, $C(t) = t - S_{N(t)}$, $t > 0$ and $D(t) = T(t) + C(t)$, $t > 0$. $T$ is called excess time, $C$ current life time and $D$ life time. Now let $N$ be a Poisson process with intensity $\lambda > 0$.

(a) Calculate the distribution of $T(t)$.

(b) Show, that the distribution of the current life time is given by

$$P(C(t) \leq s) = \exp(-\lambda t) \delta(s) + \int_0^t f_{C(t)N(t)>0}(x) dx, \quad s \in [0,t]$$

with $f_{C(t)N(t)>0}(x) = \lambda \exp(-\lambda x) 1_{\{x \leq t\}}$.

(c) Show that $P(D(t) \leq x) = (1 - (1 + \lambda \min\{t,x\}) \exp(-\lambda x)) 1_{\{x \geq 0\}}$.

(d) To determine $ET(t)$ consider the following argumentation: On average $t$ lies half between $S_{N(t)}$ and $S_{N(t)+1}$, i.e. $ET(t) = \frac{1}{2} E(S_{N(t)+1} - S_{N(t)}) = \frac{1}{2} ET_{N(t)+1} = \frac{1}{2t}$. From part (a) we know that this is not true. Why?

Exercise 2 (4 Points)
Let $X = \{ X(t), t \geq 0 \}$ be a compound Poisson process,

$$X(t) = \sum_{i=1}^{N(t)} U_i, \quad t \geq 0$$

where $\{ U_i \}_{i \in \mathbb{N}}$ is a sequence of non-negative i.i.d. random variables which are independent of $N$, and $N$ is a homogeneous Poisson process. Show that the Laplace transform of $X(t)$ is given by

$$\hat{I}_{X(t)}(s) = m_{N(t)}(\hat{I}_v(s)), \quad s \geq 0$$

where $m_{N(t)} = E e^{N(t)}$ denotes the generating function of $N(t)$.

Exercise 3 (3 Points)
Let $X = \{ X(t), t \geq 0 \}$ be a compound Poisson process like in Exercise 2, with $U_1 \sim \text{Exp}(\gamma)$. Show, that

$$\hat{I}_{X(t)}(s) = \exp\left(-\frac{\lambda s}{\gamma + s}\right), \quad s \geq 0$$

where $\lambda > 0$ denotes the intensity of $N(t)$.

Exercise 4 (2 Points)
Consider a Poisson counting measure $N$ as in Exercise 4, Sheet 5. Let $B_0 \in \mathcal{B}(\mathbb{R}^d)$ and $\tilde{\mu}(\cdot) := \mu(\cdot \cap B_0)$. Show, that the random counting measure $\tilde{N} = \{ \tilde{N}(B), B \in \mathcal{B}(\mathbb{R}^d) \}$ defined by $\tilde{N}(\cdot) := N(\cdot \cap B_0)$ is a Poisson counting measure with intensity measure $\tilde{\mu}$.