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WS 2012/2013

Stochastics II
Exercise Sheet 8

Due to: Wednesday, 12th of December 2012

Exercise 1 (4 Points)

An approximation for the Wiener process $W = \{W(t),\ t \in [0,1]\}$ is given by

$$W_n(t) = \sum_{k=1}^{n} S_k(t) Z_k \tag{1}$$

where S_k are the Schauder functions, $t \in [0,1]$, $k \geq 1$ and $Z_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$. The convergence of the sequence in (1) ($n \to \infty$) has to be understood in L_2-sense for all $t \in [0,1]$. Show that

$$W_n(t) \xrightarrow{L_2} W(t) \quad (n \to \infty)$$

Exercise 2 (4 Points)

Let $W = \{W(t),\ t \in [0,\infty]\}$ be a Wiener process. Show that the following processes are Wiener processes as well

$$W_1(t) = \begin{cases} 0 & ,\ t = 0 \\ tW(\frac{1}{t}) & ,\ t > 0 \end{cases} \qquad W_2(t) = \sqrt{c}W(\tfrac{t}{c}), \quad c > 0$$

Exercise 3 (8 Points)

Let $W = \{W(t),\ t \in [0,\infty]\}$ be a Wiener process. Define the process of the maximum as $M = \{M(t) := \max\limits_{s \in [0,t]} W(s),\ t \geq 0\}$. Show:

(a) The probability densitiy of $M(t)$ is given by

$$f_{M(t)}(x) = \sqrt{\frac{2}{\pi t}} \exp\left(-\frac{x^2}{2t}\right) \mathbb{1}\{x \geq 0\}$$

Hint: Use the fact that $P(M(t) > x) = 2P(W(t) > x)$.

(b) The expectation and the variance of $M(t)$ are given via

$$EM(t) = \sqrt{\tfrac{2t}{\pi}} \qquad VarM(t) = t(1 - \tfrac{2}{\pi})$$

(c) Let $\tau(x) := \min\{s \in \mathbb{R},\ W(s) = x\}$ be the first time when W attains the value x. Calculate the density of $\tau(x)$ and show that $E\tau(x) = \infty$.