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WS 2012/2013

Stochastics II
Exercise Sheet 9

Due to: Wednesday, 19th of December 2012

Exercise 1 (3 Points)

Let $W = \{W(t),\ t \geq 0)\}$ be the Wiener process. Define

$$Q(a,b) = P(\exists\ t \geq 0,\ W(t) > at + b)$$

$a, b > 0$. $Q(a,b)$ is the probability that W crosses the half-line $y = at + b$. Show the following statements:

(a) $Q(a,b) = Q(b,a)$ and $Q(a, b_1 + b_2) = Q(a, b_1)Q(a, b_2)$

(b) $Q(a,b) = \exp(-2ab)$

Exercise 2 (3 Points)

Let $X_1, \ldots, X_n$ be a sequence of i.i.d. infinitely divisible (ID) random variables. Show that the random variable Y given by

$$Y = \sum_{k=1}^{n} a_i X_i$$

is ID as well for arbitrary $a_1, \ldots, a_n \in \mathbb{R}$ **if the Lévy measure of X_1 is absolutely continuous w.r.t. the Lebesgue measure** on $\mathbb{R}$. Calculate the Lévy characteristics of Y.

Exercise 3 (6 Points)

Let X be a random variable with characteristic function φ.

(a) Show: If X is ID $\Rightarrow$ $\varphi(z) \neq 0, \forall\ z \in \mathbb{R}$

Hint: Show that $\lim_{n\to\infty} \varphi_n(z) = 1$ for all $z \in \mathbb{R}$ and $\varphi(z) = (\varphi_n(z))^n$. Use the fact that $|\varphi_n|^2$ is a characteristic function.

(b) Find an example for a random variable which is not ID (with proof).

Exercise 4 (4 Points)

Show that the function $\varphi : \mathbb{R} \to \mathbb{C}$ given by

$$\varphi(z) = \exp(\psi(z)), \qquad \psi(z) = 2 \sum_{k=-\infty}^{\infty} 2^{-k}(\cos(2^k z) - 1)$$

is the characteristic function of an ID random variable.

Hint: Use the Lévy-Kchinchin representation with Lévy measure $\nu(\{\pm 2^k\}) = 2^{-k}$, $k \in \mathbb{Z}$