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Stochastics II
Exercise Sheet 1

Due to: Wednesday, 24th of October 2012

In the following we use the notation from the lecture notes. For a random function $X = \{X(t),\ t \in T\}$ associated with $(\mathcal{S}_t, \mathcal{B}_t)_{t\in T}$ let $\mathcal{S}_{t_1,\dots,t_n} = \mathcal{S}_{t_1} \times \cdots \times \mathcal{S}_{t_n}$ as well as $\mathcal{B}_{t_1,\dots,t_n} = \mathcal{B}_{t_1} \otimes \cdots \otimes \mathcal{B}_{t_n}$ where $n \in \mathbb{N}$ and $t_1, \dots, t_n \in T$. All random elements are defined on a common probability space $(\Omega, \mathcal{A}, P)$.

Exercise 1 (4 Points)

Show the following statement: A family of probability measures $P_{t_1,\dots,t_n}$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, $n \geq 1$, $t = (t_1, \dots, t_n)^\top \in T^n$ fulfills the conditions of the Theorem of Kolmogorov iff for all $n \geq 2$ and all $s = (s_1, \dots, s_n)^\top \in \mathbb{R}^n$ the following conditions hold

(a) $\varphi_{P_{t_1,\dots,t_n}}((s_1, \dots, s_n)^\top) = \varphi_{P_{t_{\pi(1)},\dots,t_{\pi(n)}}}((s_{\pi(1)}, \dots, s_{\pi(n)})^\top)$ for all $\pi \in \mathrm{Perm}_n$.

(b) $\varphi_{P_{t_1,\dots,t_{n-1}}}((s_1, \dots, s_{n-1})^\top) = \varphi_{P_{t_1,\dots,t_n}}((s_1, \dots, s_{n-1}, 0)^\top)$.

Remark: $\varphi(\cdot)$ denotes the characteristic function of the corresponding measures. Perm_n denotes the group of all permutations $\pi : \{1, \dots, n\} \to \{1, \dots, n\}$.

Exercise 2 (4 Points)

Show the existence of a random function with finite dimensional multivariate Gaussian distributions and specify the measurable spaces $(\mathcal{S}_{t_1,\dots,t_n}, \mathcal{B}_{t_1,\dots,t_n})$.

Exercise 3 (4 Points)

Let $X = \{X(t),\ t \in T\}$ and $Y = \{Y(t),\ t \in T\}$ be two stochastic processes on a common probability space $(\Omega, \mathcal{A}, P)$ with values in $(\mathcal{S}, \mathcal{B})$.

(a) Show that if X and Y are stochastically equivalent then they have the same distribution, i.e. $P_X = P_Y$.

(b) Give an example of two stochastic processes which are not equivalent but have the same distribution.

Exercise 4 (4 Points)

Let $X = \{X(t),\ t \in T\}$ be a random function. Show that the vector $(X(t_1), \dots, X(t_n))^\top$ is $\mathcal{A}|\mathcal{B}_{t_1,\dots,t_n}$-measurable for every $n \in \mathbb{N}$ and arbitrary $t_1, \dots, t_n \in T$.

Hint: It holds $(X(t_1, \omega), \dots, X(t_n, \omega))^\top = p_{t_1,\dots,t_n} \circ X(\omega)$ with the projection map $p_{t_1,\dots,t_n} : \{f,\ f(t) \in \mathcal{S}_t,\ t \in T\} \to \mathcal{S}_{t_1,\dots,t_n},\ f \mapsto (f(t_1), \dots, f(t_n))^\top$.