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Stochastics II
Exercise Sheet 3

Due to: Wednesday, 7th of November 2012

Exercise 1 (3 Points)

Let $X = \{X(t),\ t \in T\}$, with $T \subset \mathbb{R}$ compact, be a real-valued separable stochastic process on a **complete** probability space $(\Omega, \mathcal{A}, P)$. Show that $\sup_{t \in T} X(t)$ is a random variable, i.e. a $\mathcal{A}|\mathcal{B}(\mathbb{R})$-measurable mapping.

Exercise 2 (5 Points)

Let $X = \{X(t),\ t \in [a, b]\}$ be a real-valued stochastic process. One can show, that X has a continuous modification if

$$E|X(t+h) - X(t)|^{\alpha} \leq C|h|^{1+\epsilon} \tag{1}$$

for sufficient small h and for some $\alpha,\ \epsilon > 0$ and a constant $C := C(\alpha, \epsilon) > 0$. Show the following statements:

(a) In general the condition above does not imply the existence of a continuous modification if (1) holds for $\epsilon = 0$.
Hint: Consider the Poisson-process.

(b) The Wiener-process has a continuous modification.
Hint: Use $\alpha = 4$.

Exercise 3 (4 Points)

Show that the Wiener-process is not stochastically differentiable at every point $t \in [0, \infty)$.

Exercise 4 (4 Points)

For a complex-valued stochastic process $X = \{X(t), t \in T\}$ the covariance function is defined as

$$C(s,t) = E\left[(X(s) - EX(s))\overline{(X(t) - EX(t))}\right], \qquad s, t \in T$$

where $\overline{z}$ denotes the conjugate complex of $z \in \mathbb{C}$. Show that $C(s,t)$ has the following properties:

(a) C is symmetric, i.e. $C(s,t) = \overline{C(t,s)}$.

(b) $C(t,t) = Var(X(t))$, $t \in T$.

(c) C is positiv semidefinite, i.e. for all $n \in \mathbb{N}$, $t_1, \dots, t_n \in T$ and $z_1, \dots, z_n \in \mathbb{C}$ it holds

$$\sum_{i=1}^{n} \sum_{j=1}^{n} C(t_i, t_j) z_i \overline{z_j} \geq 0$$