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Stochastics III - Exercise sheet 2

Due to: 7. 11. 2012 before exercises start

Exercise 1 (3 points)
Give an example of two random variables X and Y which are normally distributed but such that the vector $(X,Y)^\top$ is not multivariate normally distributed.

Exercise 2 (2 + 3 points)
Let $\mathbf{X} = (X_1, X_2, X_3)^\top \sim \mathrm{N}(\boldsymbol{\mu}, \mathbf{K})$ with expectation vector $\boldsymbol{\mu} = (1,2,3)^\top$ and covariance matrix

$$\mathbf{K} = \sigma^2 \begin{pmatrix} 2 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 2 \end{pmatrix}.$$

(a) Determine the marginal distributions of X_2 und $(X_1, X_3)^\top$.

(b) For which ρ are the random variables $X_1 + X_2 + X_3$ and $X_1 - X_2 - X_3$ independent?

Exercise 3 (4 + 4 points)

Proof Theorem 1.4 of the lecture course:
Let $\mathbf{Y}$ be an n-dimensional random vector with expectation vector $\boldsymbol{\mu} = \mathbb{E}\mathbf{Y}$ and covariance matrix $\mathbf{K} = \mathrm{Cov}(\mathbf{Y})$, such that $\mathrm{rk}(\mathbf{K}) = r$ with $r \le n$. The random vector $\mathbf{Y}$ is normally distributed if and only if one of the following conditions is fulfilled.

(a) The characteristic function $\varphi(\mathbf{t}) = \mathbb{E}\exp\Big(\mathrm{i}\sum_{j=1}^n t_j Y_j\Big)$ of $\mathbf{Y}$ is given by

$$\varphi(\mathbf{t}) = \exp\Big(\mathrm{i}\mathbf{t}^\top\boldsymbol{\mu} - \frac{1}{2}\,\mathbf{t}^\top\mathbf{K}\mathbf{t}\Big), \qquad \forall\, \mathbf{t} = (t_1, \ldots, t_n)^\top \in \mathbb{R}^n.$$

(b) The linear function $\mathbf{c}^\top\mathbf{Y}$ of $\mathbf{Y}$ for every $\mathbf{c} \in \mathbb{R}^n$ with $\mathbf{c} \ne \mathbf{o}$ is normally distributed with

$$\mathbf{c}^\top\mathbf{Y} \sim \mathrm{N}(\mathbf{c}^\top\boldsymbol{\mu}, \mathbf{c}^\top\mathbf{K}\mathbf{c}).$$

Exercise 4 (6 + 2 points)
Let $(X,Y)^\top$ bivariate normally distributed with expectation vector $\boldsymbol{\mu} = (1,2)^\top$ and covariance matrix

$$\mathbf{K} = \begin{pmatrix} 5 & 2 \\ 2 & 3 \end{pmatrix}.$$

(a) Plot with **R** the density $f_{(X,Y)}$ of the random vector $(X,Y)^\top$. Hint: Use the commands *dmvnorm, persp.*

(b) Calculate in **R** the expectation and the variance of the random variable $2X - Y + 3$.