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Stochastics III - Exercise sheet 3

Due to: 28. 11. 2012 before exercises start

Exercise 1 (3 + 3 points)

(a) Let $\boldsymbol{Z} = (Z_1, Z_2, Z_3)^\top \sim N(\boldsymbol{\mu}, \boldsymbol{I}_3)$ with expectation vector $\boldsymbol{\mu} = (1, 7, -5)^\top$ and covariance matrix $\boldsymbol{I}_3$. Determine the distribution of

$$\frac{1}{2}Z_1^2 + Z_2^2 + \frac{1}{2}Z_3^2 - Z_1 Z_3 \,.$$

(b) Let $\boldsymbol{Z} = (Z_1, Z_2, Z_3)^\top \sim N(\boldsymbol{\mu}, \boldsymbol{K})$ with expectation vector $\boldsymbol{\mu} = (1, -3, 2)^\top$ and covariance matrix

$$\boldsymbol{K} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Determine the distribution of

$$Z_1^2 + Z_2^2 + Z_3^2 + 2Z_1Z_2 - 2Z_1Z_3 - 2Z_2Z_3 \,.$$

Exercise 2 (3 points)

Reconsider the two-dimensional normally distributed random vector $(X, Y)^\top$ of exercise sheet 2 / exercise 4, i.e. $(X, Y)^\top \sim N(\boldsymbol{\mu}, \boldsymbol{K})$ with

$$\boldsymbol{\mu} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{K} = \begin{pmatrix} 5 & 2 \\ 2 & 3 \end{pmatrix} .$$

Draw a histogram based on 10000 realisations of $(X, Y)^\top$ and compare it with the density function in exercise sheet 2 / exercise 4.

Exercise 3 (6 points)

Proof Lemma 1.11 of the lecture course.
Let $Z = (Z_1, \ldots, Z_n)^\top \sim N(o, K)$ a normally distributed random vector with covariance matrix $K = (k_{ij})_{i,j=1,\ldots,n}$. Show that for any $i, j, l, m \in \{1, \ldots, n\}$ it holds

$$\mathbb{E}\,(Z_i Z_j Z_l) = 0$$

and

$$\mathbb{E}\,(Z_i Z_j Z_l Z_m) = k_{ij}k_{lm} + k_{il}k_{jm} + k_{jl}k_{im}.$$

Exercise 4 (6 points)
Let $X = (X_1, \ldots, X_n) \sim N(\boldsymbol{\mu}, \boldsymbol{I_n})$ be multivariate normally distributed with expectation vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{I_n}$. Determine the characteristic function and the expectation of $X^\top X$.