Report of the Seminar „Characteristic functions and infinitely divisible distributions“

Joung In Kim

March 8, 2013

1 Two different characteristic functions can be the same on a finite interval but not on the whole real line

If there exist two characteristic functions ϕ_1 and ϕ_2 which are coincided in the interval $[-l, l]$, are these functions also the same on the whole real line $\mathbb{R}$? Let us consider the following example :

$$h(x) = \begin{cases} 0, & \text{if } |x| > \pi/2 \\ x, & \text{if } |x| \leq \pi/2 \end{cases}$$

We define c(t)=$\int_{-\infty}^{\infty} h(x)h(x+t)dx$. Then

$$\phi_1(t) = \frac{c(t)}{c(0)} = \begin{cases} 1 + 3\pi^{-1}t - 2\pi^{-3}t^3, & \text{if } -\pi \leq t \leq 0 \\ 1 - 3\pi^{-1}t + 2\pi^{-3}t^3, & \text{if } 0 \leq t \leq \pi \\ 0, & \text{if } |t| > \pi \end{cases}$$

is a characteristic function, because $\phi_1(0) = \frac{c(0)}{c(0)} = 1$, and $\phi_1(t)$ is positive definite.

Furthermore, we can define $\phi_2(t)$ as follows :

$$\phi_2(t) = \phi_1(t), \text{ for } |t| \leq \pi$$
$$\phi_2(t + 2\pi) = \phi_2(t), \text{ for } t \in \mathbb{R}$$

Now, we will prove that $\phi_2(t)$ is a characteristic function. Then we can obtain characteristic functions which coincide in $[-\pi, \pi]$ but not on the whole $\mathbb{R}$ and our proof

will be completed.
By using Fourier Transform, we can describe that $\phi_2(t)$ is a characteristic function. Because $\phi_2(t)$ is an even function, its Fourier expansion must be of the form

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nt) \tag{1}$$

$$\text{where } a_0 = \frac{2}{\pi}\int_{-\pi}^{\pi} \phi_2(t)dt,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi_2(t)\cos(nt)dt$$

If we calculate the coefficients, it can be illustrated :

$$a_0 = 0,\ a_n = 6\pi^{-2}[n^{-2}(1+\cos(nt)\pi) + 4\pi^{-2}n^{-4}(1-\cos(n\pi))],\ n = 1, 2, \ldots$$

It is easily proved that the series in (1) converges uniformly, all of the coefficients are non-negative and the sum of the coefficients is equal to 1.
Proof. (i) $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nt) \leq \frac{1}{2}a_0 + \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} a_n = 1$. Therefore, series converges uniformly. (ii) $1 + \cos(n\pi)$ have values between 0 and 2. Accordingly all of the coefficients are non-negative. (iii) $\sum_{n=1}^{\infty} a_n = \phi_2(0) = 1$.□

Remark 1.1. *If $\phi_n(t)$ are positive definite for each n then so is any linear combination $\phi(t) = \sum_n a_n\phi_n(t)$ with non-negative weights a_n. If each $\phi_n(t)$ is normalized with $\phi_n(0) = 1$ and $\sum_n a_n = 1$, then of of course $\phi(0) = 1$ as well.*

Now let $\phi_n(t) = \cos(nt)$. Then $\phi_n(0) = 1$ and $\phi_n(t)$ is positive definite for all $n \in \mathbb{N}$.
Proof. (i) $\phi_n(0) = \cos(0) = 1$.
(ii) $\phi_n(t) = \cos(nt) = 0.5(e^{itn} + e^{-itn})$ is the characteristic function of the probability distribution which take values n and $-n$ with probability 0.5 respectively. Therefore, it is positive definite. □

According to the remark 1.1 $\phi_2(0) = 1$ and $\phi_2(t) = \sum_{n=1}^{\infty} a_n \cos(nt)$ is positive definite. Therefore, ϕ_2 is a characteristic function. Consequently, we can conclude that $\phi_1(t)$ and $\phi_2(t)$ are not always coincide for all t $\in \mathbb{R}$ although $\phi_1(t) = \phi_2(t)$ for t $\in [-\pi, \pi]$.

Based on the result above, it is observable the following interesting phenomenon. Let say F_1 and F_2 are distribution functions of $\phi_1(t)$ and $\phi_2(t)$, respectively. Then the following equation will be satisfied.

$$\phi_1(t)\phi_1(t) = \phi_1(t)\phi_2(t) \text{ for all } t \in \mathbb{R}$$

This equation can be replaced by distribution functions F_1 and F_2:

$$F_1 * F_1 = F_1 * F_2$$

However the calcellation law will not be satisfied here. This is, $F_1 = F_2$ is not valid. As a consequence, we can recognize that the quotient of two characteristic functions will not be always a characteristic function. The exact example with this phenomenon will be illustrated in the following section.

2 Is the ratio of two characteristic functions also a characteristic function?

Assume that there are two characteristic functions ϕ_1 and ϕ_2. And the interesting point is now whether or not the ratio ϕ_1/ϕ_2 is also a characteristic function. Through the counterexamle, it is allowable to see that it is not always true. Before illustrating the example, let us introduce both the definition of an analytic characteristic function and one of the properties of such a function.
A characteristic function ϕ is said to be analytic if there exists a number $r > 0$ such that ϕ can be expressed with a convergent power series in the interval $(-r, r)$, i.e. $\phi(t) = \sum_{k=0}^{\infty} a_k t^k/k!, t \in (-r, r)$, with some complex coefficients a_k.

Corollary 2.1. *A necessary condition that a function, which is analytic in some neighbourhood of the origin, be a characteristic function, is that in either half-plane the singularity nearest to the real axis be located on the imaginary axis.*[1]

This corollary will be used to prove whether or not the following functions are characteristic functions. Consider the following functions :

$$\phi_1(t) = \left[\left(1 - \frac{it}{a}\right)\left(1 - \frac{it}{a+ib}\right)\left(1 - \frac{it}{a-ib}\right)\right]^{-1}, \ \phi_2(t) = \left(1 - \frac{it}{a}\right)^{-1}, t \in \mathbb{R}$$

where $a \geq b > 0$. The functions ϕ_1 and ϕ_2 are both analytic and have singularities at $-ai, -ai+b$ and $-ai-b$ for ϕ_1 and $-ai$ for ϕ_2, respectively. Therefore, based on the corollary 2.1, both of them are characteristic functions. Now let $\psi(t)$ be the ratio of them, that is,

$$\psi(t) = \frac{\phi_1(t)}{\phi_2(t)} = \left[\left(1 - \frac{it}{a+ib}\right)\left(1 - \frac{it}{a-ib}\right)\right]^{-1}$$

$\psi(t)$ has singularities at both $-ai+b$ and $-ai-b$. And both of the singularities are not located on the imaginary axis. Therefore, it leads to figure out that $\psi(t)$ is not a characteristic function and we have proved that the ratio of two chariceristic function is in general not a characteristic function.

3 If a distribution function F is absolutely continuous, is the characteristic function absolutely integrable?

One of the properties of a characteristic function is that, if a characteristic function ϕ is absolutely integrable on $\mathbb{R}$, then the distribution function F is also absolutely continuous

[1] see E. Lukacs. 1970, p.193.

and it's density $f = F'$ can be calculated from the inverse Fourier transform of ϕ. In this example, it will be proved that the converse statement is not true. To do that, we will introduce a theorem, which is called the "Theorem of G. Pólya".

Theorem 3.1 (*Theorem of G.Pólya.*). *If $\psi(t), t \in \mathbb{R}$ is a real-valued continuous function, which satisfies following conditions:*
(i) $\psi(0)=1$
(ii) $\psi(-t) = \psi(t)$
(iii) $\psi(t)$ is convex for t>0, that is, $\psi(\frac{t_1+t_2}{2}) \leq \frac{1}{2}(\psi(t_1) + \psi(t_2))$ for all $t_1 > 0, t_2 > 0$
(iv) $\lim_{t\to\infty} \psi(t) = 0$
Then ψ is a characteristic function of an absolutely continuous distribution.[2]

Let us discuss the following examples.

$$\psi_1(t) = \frac{1}{1+|t|}, \text{ if } t \in \mathbb{R}$$

$$\psi_2(t) = \begin{cases} 1-|t| & \text{, if } 0 \leq |t| \leq \frac{1}{2} \\ \frac{1}{4|t|} & \text{, if } |t| \geq \frac{1}{2}. \end{cases}$$

It can be proved that these two functions satisfy all of the four conditions of the Theorem 3.1.

Proof. (i) $\psi_1(0) = \frac{1}{1+|0|} = 1, \ \psi_2(0) = 1 - |0| = 1.$
(ii) $|-t| = |t| => \psi_1(-t) = \psi_1(t), \ \psi_2(-t) = \psi_2(t).$
(iii) $\psi_1'(t) = -\frac{1}{(1+t)^2}$ and $\psi_1''(t) = \frac{2}{(1+t)^3} \geq 0$ for $t > 0$. Hence $\psi_1(t)$ is convex for $t > 0$
$\psi_2'(t) = -1$ for $0 \leq t \leq \frac{1}{2}$ and $-\frac{1}{4t^2}$ for $t \geq \frac{1}{2}$, $\psi_2''(t) = 0$ for $0 \leq t \leq \frac{1}{2}$ and $\frac{1}{2t^3}$ for $t \geq \frac{1}{2}$. So $\psi_2''(t) \geq 0$ and $\psi_2(t)$ is convex for all $t > 0$.
(iv) $\lim_{t\to\infty} \psi_1(t) = \lim_{t\to\infty} \frac{1}{1+|t|} = 0, \ \lim_{t\to\infty} \psi_2(t) = \lim_{t\to\infty} \frac{1}{4|t|} = 0.\square$

Therefore, we can conclude that they are characteristic functions of absolutely continuous distributions although they are not absolutely integrable. In conclusion, it means that the converse statement is not true.

4 Absolute value of a characteristic function $|\phi|$ can be an infinitely divisible characteristic function, although ϕ is not infinitely divisible

Definition 4.1. *Let X be a random variable with a distribution function F and a characteristic function ϕ. Then X as well as F and ϕ are infinitely divisible if for each*

[2]for the proof see E. Lukacs. 1970, p. 83-84.

$n \geq 1$ there exists independently identically distributed random variables $X_{n1}, \ldots, X_{nn}$ such that

$$X = X_{n1} + \ldots + X_{nn}$$

or equivalently,

$$F = F_n * \cdots * F_n = (F_n)^{*n} \text{ or } \phi = (\phi_n)^n.$$

Remark 4.1. *A characteristic function of an infinitely divisible random variable can be expressed in following Lévy-Chintschin-Formel.*

$$\phi(t) = exp\left\{i\gamma t + \int_{-\infty}^{\infty}\left(e^{itu} - 1 - \frac{itu}{1+u^2}\right)\frac{1+u^2}{u^2}dG(u)\right\}$$

where $\gamma \in \mathbb{R}$, and $G(x), x \in \mathbb{R}$, is a non-decreasing left-continuous function of bounded variation and $G(-\infty) = 0$.

One of the properties of an infinitely divisible characteristic function is followings. If $\phi(t)$ is an infinitely divisible characteristic function, then $|\phi(t)|$ is also an infinitely divisible characteristic function.

Proof. Let $\phi(t)$ be an infinitely divisible characteristic function. Then $\phi(-t)$ is also an infinitely divisible characteristic function. It follows that $|\phi(t)|^2$ is an infinitely divisible characteristic function because $|\phi(t)|^2 = \phi(t)\overline{\phi}(t) = \phi(t)\phi(-t)$ and the product of a finite number of infinitely divisible characteristic function is also infinitely divisible. Consequently $(|\phi(t)|^2)^{\frac{1}{2n}} = |\phi(t)^{\frac{1}{n}}|$ is a characteristic function. It means that for every integer n, there exists $|\phi(t)^{\frac{1}{n}}|$ such that $|\phi(t)| = [|\phi(t)^{\frac{1}{n}}|]^n$. Hence, $|\phi(t)|$ is an infinitely divisible characteristic function.□

However, generally speaking, the converse statment is not true. It can be proved with the following counterexample. That is, $|\phi|$ is infinitely divisible, but ϕ is a characteristic function which is not infinitely divisible.

Consider the function

$$\phi(t) = \left[\frac{1-b}{1-a}\right]\left[\frac{1-ae^{-it}}{a-be^{it}}\right], t \in \mathbb{R}$$

where $0 < a \leq b < 1$. By applying the formula for a geometric series, it can be transformed like

$$\phi(t) = \left[\frac{1-b}{1-a}\right]\left[-ae^{-it} + (1-ab)\sum_{k=0}^{\infty} b^k e^{itk}\right]$$

It is easy to see that ϕ is the characteristic function of a random variable X with the following distribution:

$$\mathbb{P}[X=-1] = -a\left[\frac{1-b}{1-a}\right],\ \mathbb{P}[X=k] = \left[\frac{1-b}{1-a}\right](1-ab)b^k,\ k=0,1,2,\dots$$

Let's look whether ϕ is infinitely divisible or not. It is able to find that

$$\log(\phi(t)) = \sum_{k=1}^{\infty}[(-1)^{k-1}k^{-1}a^k(e^{-itk}-1) + b^k k^{-1}(e^{itk}-1)]$$

Then, $\log(\phi(t))$ can be written in the Lévy-Khintchine representation with

$$\gamma = \sum_{k=1}^{\infty}\frac{b^k + (-1)^k a^k}{k^2+1}$$

Also, $G(x)$ can be a function of bounded variation with jumps of size $kb^k/(k^2+1)$ at $x=k$ and $(-1)^{k-1}ka^k/(k^2+1)$ at $x=-k$ for $k=1,2,\dots$. In this case, G is not monotone, which means that ϕ is not infinitely divisible.

It is possible to see that the function

$$\overline{\phi}(t) = \left[\frac{1-b}{1-a}\right]\left[\frac{1-ae^{it}}{a-be^{-it}}\right]$$

is also a characteristic function, which is not infinitely divisible.
Now we are focusing on whether or not the following function is infinitely divisible.

$$\psi(t) = |\phi(t)|^2 = \phi(t)\overline{\phi}(t)$$

Note that

$$\log(\psi(t)) = \sum_{k=1}^{\infty}k^{-1}[b^k+(-1)^{k-1}a^k](e^{-itk}-1) + \sum_{k=1}^{\infty}k^{-1}[b^k+(-1)^{k-1}a^k](e^{itk}-1)$$

We have $\gamma=0$ and $G(x)$ is a function with jumps of size $k(k^2+1)^{-1}[b^k+(-1)^{k-1}a^k]$ at the points $x=\pm k, k=1,2,\dots$ in the Lévy-Khintchine representation. Since $G(x)$ is a non-decreasing function, ψ is infinitely divisible. Furthermore, $|\phi| = (|\phi|^2)^{1/2}$ is also infinitely divisible.

References

[1] J. Stoyanov. *Counterexamples in probability* (2nd edition). Wiley, 1997

[2] K. L. Chung. *A course in probability theory.* Academic Press, 1974

[3] E. Lukacs. *Characteristic functions.* Griffin, 1970