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Lisa Handl Winter Term 2013 / 2014

Practice Exam

Instructions

- Give as much detail as you can.
- If you are asked to describe an algorithm, write out all the steps and include all the information you have that is specific to the problem.

Exercise 1

a) Write out an algorithm to generate standard uniform pseudo-random numbers using a multiple-recursive random number generator.

b) Assume you have an implementation of a multiple-recursive generator and want to sample from the density

$$f(x) = \begin{cases} \frac{1}{2x}, & \text{if } x \in [1, \mathrm{e}] \\ \frac{1}{2}, & \text{if } x \in [4, 5] \\ 0, & \text{otherwise.} \end{cases}$$

Give an algorithm to do this based on the inverse-transform method.

c) Describe a method you could use instead of inverse-transform in b). Which would you prefer in this case and why?

Exercise 2

A fair die is rolled repeatedly. Consider the following sequences of random variables:

1. The largest number X_n in the first n rolls.
2. The second largest number Y_n in the first n rolls.
3. The number N_n of sixes in the first n rolls.
4. The time (measured at the nth step) C_n since the most recent 6.

Answer the following questions:

a) Which of the sequences are Markov chains?

b) For each Markov chain: write out the transition matrix.

c) For each Markov chain: wtate if it is irreducible.

d) For each Markov chain: state if it is recurrent.

e) For each Markov chain: state if it is aperiodic.

Exercise 3

A flea jumps around on the four corners of a square. It can only jump horizontally and vertically (so it can jump to one of two neighbouring corners at each jump). It jumps to either one of these corners with equal probability.

a) Write out the transition matrix, P, of this Markov Chain.

b) Clearly this chain is reversible. Find the stationary distribution π by solving the detailed balance equations.

c) If $X_n \sim \pi$ what is the distribution of X_{n+1}?

d) If the flea always starts in the top left corner, and the chain is run for a very long time, will the distribution of the chain converge to the stationary distribution?

Exercise 4

a) Write an algorithm to sample (approximately) from some density $f(x)$ using the Metropolis-Hastings independence sampler with the proposal density $g(x)$.

b) Choose a proposal density to sample from the Beta(p, q) distribution with parameters $p, q \geq 1$ and $p + q > 2$? The Beta distribution has the density

$$f(x) \propto x^{p-1}(1-x)^{q-1}\mathbb{1}\{0 \leq x \leq 1\}.$$

Calculate the acceptance probabilities $\alpha(x, y)$ for $x, y \in [0, 1]$.

c) Indicate an advantage of this method compared to the acceptance-rejection method.

Exercise 5

a) Assume you want to estimate ℓ, where

$$\ell = \int_0^1 \mathrm{e}^{x^2}\,\mathrm{d}x.$$

Show that using the estimator $\hat{\ell}_1 = \mathrm{e}^{U^2}\frac{\left(1+\mathrm{e}^{1-2U}\right)}{2}$ with $U \sim U(0, 1)$ is better than using the estimator $\hat{\ell}_2 = \frac{\mathrm{e}^{U_1^2}+\mathrm{e}^{U_2^2}}{2}$ with $U_1, U_2 \sim U(0, 1)$ i.i.d..

Hint: You may use without proof that

$$\mathbb{E}\left(\mathrm{e}^{2U^2-2U+1}\right) < \ell^2.$$

b) Explain one alternative technique for variance reduction and how you could apply it to estimate ℓ.

Exercise 6

In a knapsack problem you consider a set of n objects for which you know the weights $w_1, \ldots, w_n$ and the values $v_1, \ldots, v_n$. The goal is to pack a number of objects into a knapsack (backpack), such that the total value of things in the rucksack is as large as possible, while the total weight doesn't exceed a given limit w_{max}. Give a simulated annealing algorithm to solve this problem. Make sure you define an appropriate cost function S and explain how you include the weight limit.