2.7. Recurrence and transience

Consider a Markov chain $\{X_n : n \in \mathbb{N}_0\}$ on state space E with transition matrix P.

DEFINITION 2.7.1. A state $i \in E$ is called *recurrent* if

$$\mathbb{P}_i[X_n = i \text{ for infinitely many } n] = 1.$$

DEFINITION 2.7.2. A state $i \in E$ is called *transient* if

$$\mathbb{P}_i[X_n = i \text{ for infinitely many } n] = 0.$$

A recurrent state has the property that a Markov chain starting at this state returns to this state infinitely often, with probability 1. A transient state has the property that a Markov chain starting at this state returns to this state only finitely often, with probability 1.

The next theorem is a characterization of recurrent/transient states.

THEOREM 2.7.3. *Let $i \in E$ be a state. Denote by f_i the probability that a Markov chain which starts at i returns to i at least once, that is*

$$f_i = \mathbb{P}_i[\exists n \in \mathbb{N} : X_n = i].$$

Then,

(1) *The state i is recurrent if and only if $f_i = 1$.*
(2) *The state i is transient if and only if $f_i < 1$.*

COROLLARY 2.7.4. *Every state is either recurrent or transient.*

PROOF. For $k \in \mathbb{N}$ consider the random event

$$B_k = \{X_n = i \text{ for at least } k \text{ different values of } n \in \mathbb{N}\}.$$

Then, $\mathbb{P}_i[B_k] = f_i^k$. Also, $B_1 \supset B_2 \supset \ldots$. It follows that

$$\mathbb{P}_i[X_n = i \text{ for infinitely many } n] = \mathbb{P}_i[\cap_{k=1}^{\infty} B_k] = \lim_{k\to\infty} \mathbb{P}_i[B_k] = \lim_{k\to\infty} f_i^k = \begin{cases} 1, & \text{if } f_i = 1, \\ 0, & \text{if } f_i < 1. \end{cases}$$

It follows that state i is recurrent if $f_i = 1$ and transient if $f_i < 1$. □

Here is one more characterization of recurrence and transience.

THEOREM 2.7.5. *Let $i \in E$ be a state. Recall that $p_{ii}^{(n)} = \mathbb{P}_i[X_n = i]$ denotes the probability that a Markov chain which started at state i visits state i at time n. Then,*

(1) *The state i is recurrent if and only if $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$.*
(2) *The state i is transient if and only if $\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$.*

PROOF. Let the Markov chain start at state i. Consider the random variable

$$V_i := \sum_{n=1}^{\infty} \mathbb{1}_{\{X_n = i\}}$$

which counts the number of returns of the Markov chain to state i. Note that the random variable V_i can take the value $+\infty$. Then,

$$\mathbb{P}_i[V_i \geq k] = \mathbb{P}[B_k] = f_i^k, \quad k \in \mathbb{N}.$$

Thus, the expectation of V_i can be computed as follows:

$$\mathbb{E}_i[V_i] = \sum_{k=1}^{\infty} \mathbb{P}_i[V_i \geq k] = \sum_{k=1}^{\infty} f_i^k. \tag{2.7.1}$$

On the other hand,

$$\mathbb{E}_i[V_i] = \mathbb{E}_i \sum_{n=1}^{\infty} \mathbb{1}_{\{X_n = i\}} = \sum_{n=1}^{\infty} \mathbb{E}_i \mathbb{1}_{\{X_n = i\}} = \sum_{n=1}^{\infty} p_{ii}^{(n)}. \tag{2.7.2}$$

CASE 1. Assume that state i is recurrent. Then, $f_i = 1$ by Theorem 2.7.3. It follows that $\mathbb{E}_i[V_i] = \infty$ by (2.7.1). (In fact, $\mathbb{P}_i[V_i = +\infty] = 1$ since $\mathbb{P}[V_i \geq k] = 1$ for every $k \in \mathbb{N}$). Hence, $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$ by (2.7.2)

CASE 2. Assume that state i is transient. Then, $f_i < 1$ by Theorem 2.7.3. Thus, $\mathbb{E}_i V_i < \infty$ by (2.7.1) and hence, $\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$ by (2.7.2). □

The next theorem shows that recurrence and transience are class properties: If some state in a communicating class is recurrent (resp. transient), then all states in this class are recurrent (resp. transient).

THEOREM 2.7.6.

1. *If $i \in E$ be a recurrent state and $j \leftrightsquigarrow i$, then j is also recurrent.*
2. *If $i \in E$ be a transient state and $j \leftrightsquigarrow i$, then j is also transient.*

PROOF. It suffices to prove Part 2. Let i be a transient state and let $j \leftrightsquigarrow i$. It follows that there exist $s, r \in \mathbb{N}_0$ with $p_{ij}^{(s)} > 0$ and $p_{ji}^{(r)} > 0$. For all $n \in \mathbb{N}$ it holds that

$$p_{ii}^{(n+r+s)} \geq p_{ij}^{(s)} p_{jj}^{(n)} p_{ji}^{(r)}.$$

Therefore,

$$\sum_{n=1}^{\infty} p_{jj}^{(n)} \leq \frac{1}{p_{ij}^{(s)} p_{ji}^{(r)}} \sum_{n=1}^{\infty} p_{ii}^{(n+r+s)} \leq \frac{1}{p_{ij}^{(s)} p_{ji}^{(r)}} \sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty,$$

where the last step holds because i is transient. It follows that state j is also transient. □

Theorem 2.7.6 allows us to introduce the following definitions.

DEFINITION 2.7.7. A communicating class is called recurrent if at least one (equivalently, every) state in this class is recurrent. A communicating class is transient if at least one (equivalently, every) state in this class is transient.

DEFINITION 2.7.8. An irreducible Markov chain is called recurrent if at least one (equivalently, every) state in this chain is recurrent. An irreducible Markov chain is called transient if at least one (equivalently, every) state in this chain is transient.

The next theorem states that it is impossible to leave a recurrent class.

THEOREM 2.7.9. *Every recurrent communicating class is closed.*

Proof. Let C be a non-closed class. We need to show that it is not recurrent. Since C is not closed, there exist states i, j so that $i \in C$, $j \notin C$ and $i \rightsquigarrow j$. This means that there exists $m \in \mathbb{N}$ so that $p_{ij}^{(m)} = \mathbb{P}_i[X_m = j] > 0$. If the event $\{X_m = j\}$ occurs, then after time m the chain cannot return to state i because otherwise i and j would be in the same communicating class. It follows that

$$\mathbb{P}_i[\{X_m = j\} \cap \{X_n = i \text{ for infinitely many } n\}] = 0.$$

This implies that

$$\mathbb{P}_i[X_n = i \text{ for infinitely many } n] < 1.$$

Therefore, state i is not recurrent. □

If some communicating class contains only finitely states and the chain cannot leave this class, then it looks very plausible that the chain which started in some state of this class will return to this state infinitely often (and, in fact, will visit any state of this class infinitely often), with probability 1. This is stated in the next theorem.

Theorem 2.7.10. *Every finite closed communicating class is recurrent.*

Proof. Let C be a closed communicating class with finitely many elements. Take some state $i \in C$. A chain starting in i stays in C forever and since C is finite, there must be at least one state $j \in C$ which is visited infinitely often with positive probability:

$$\mathbb{P}_i[X_n = j \text{ for infinitely many } n \in \mathbb{N}] > 0.$$

At the moment it is not clear whether we can take $i = j$. But since i and j are in the same communicating class, there exists $m \in \mathbb{N}_0$ so that $p_{ji}^{(m)} > 0$. From the inequality

$$\mathbb{P}_j[X_n = j \text{ for infinitely many } n] > p_{ji}^{(m)} \cdot \mathbb{P}_i[X_n = j \text{ for infinitely many } n] > 0$$

it follows that state j is recurrent. The class C is then recurrent because it contains at leats one recurrent state, namely j. □

So, in a Markov chain with finitely many states we have the following equivalencies

(1) A communicating class is recurrent if and only if it is closed.
(2) A communicating class is transient if and only if it is not closed.

Lemma 2.7.11. *Consider an irreducible, recurrent Markov chain with an arbitrary initial distribution α. Then, for every state $j \in E$ the number of visits of the chain to j is infinite with probability* 1.

Proof. Exercise. □

2.8. Recurrence and transience of random walks

Example 2.8.1. A *simple random walk on* $\mathbb{Z}$ is a Markov chain with state space $E = \mathbb{Z}$ and transition probabilities

$$p_{i,i+1} = p, \quad p_{i,i-1} = 1 - p, \quad i \in \mathbb{Z}.$$

So, from every state the random walk goes one step to the right with probability p, or one step to the left with probability $1 - p$; see Figure 1. Here, $p \in [0, 1]$ is a parameter.

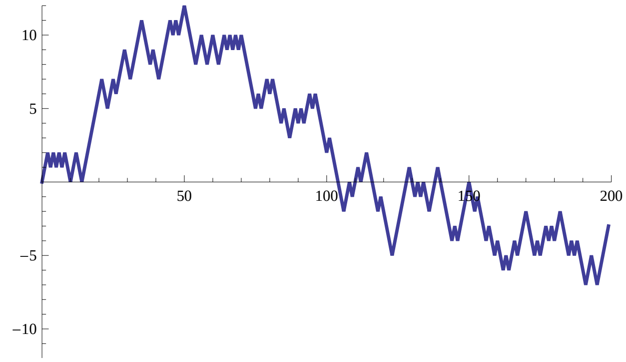

Figure 1. Sample path of a simple random walk on $\mathbb{Z}$ with $p = \frac{1}{2}$. The figure shows 200 steps of the walk.

Theorem 2.8.2. *If $p = \frac{1}{2}$, then any state of the simple random walk is recurrent. If $p \neq \frac{1}{2}$, then any state is transient.*

Proof. By translation invariance, we can restrict our attention to state 0. We can represent our Markov chain as $X_n = \xi_1 + \ldots + \xi_n$, where $\xi_1, \xi_2, \ldots$ are independent and identically distributed random variables with Bernoulli distribution:

$$\mathbb{P}[\xi_k = 1] = p, \quad \mathbb{P}[\xi_k = -1] = 1 - p.$$

Case 1. Let $p \neq \frac{1}{2}$. Then, $\mathbb{E}\xi_k = p - (1-p) = 2p - 1 \neq 0$. By the strong law of large numbers,

$$\lim_{n\to\infty} \frac{1}{n} X_n = \lim_{n\to\infty} \frac{\xi_1 + \ldots + \xi_n}{n} = \mathbb{E}\xi_1 \neq 0 \quad \text{a.s.}$$

In the case $p > \frac{1}{2}$ we have $\mathbb{E}\xi_1 > 0$ and hence, $\lim_{n\to\infty} X_n = +\infty$ a.s. In the case $p < \frac{1}{2}$ we have $\mathbb{E}\xi_1 < 0$ and hence, $\lim_{n\to\infty} X_n = -\infty$ a.s. In both cases it follows that

$$\mathbb{P}[X_n = 0 \text{ for infinitely many } n] = 0.$$

Hence, state 0 is transient.

Case 2. Let $p = \frac{1}{2}$. In this case, $\mathbb{E}\xi_k = 0$ and the argument of Case 1 does not work. We will use Theorem 2.7.5. The n-step transition probability from 0 to 0 is given by

$$p_{00}^{(n)} = \begin{cases} 0, & \text{if } n = 2k+1 \text{ odd}, \\ \frac{1}{2^{2k}}\binom{2k}{k}, & \text{if } n = 2k \text{ even}. \end{cases}$$

The Stirling formula $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$, as $n \to \infty$, yields that

$$p_{00}^{(2k)} \sim \frac{1}{\sqrt{\pi k}}, \quad \text{as } k \to \infty.$$

Since the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges, it follows that $\sum_{n=1}^{\infty} p_{00}^{(n)} = \sum_{k=1}^{\infty} p_{00}^{(2k)} = \infty$. By Theorem 2.7.5, this implies that 0 is a recurrent state. □

Example 2.8.3. The *simple, symmetric random walk on $\mathbb{Z}^d$* is a Markov chain defined as follows. The state space is the d-dimensional lattice

$$\mathbb{Z}^d = \{(n_1, \ldots, n_d) : n_1, \ldots, n_d \in \mathbb{Z}\}.$$

Let $e_1, \dots, e_d$ be the standard basis of $\mathbb{R}^d$, that is

$$e_1 = (1, 0, 0 \dots, 0), \quad e_2 = (0, 1, 0, \dots, 0), \quad e_3 = (0, 0, 1, \dots, 0), \quad \dots, \quad e_d = (0, 0, 0, \dots, 1).$$

Let $\xi_1, \xi_2, \dots$ be independent and identically distributed d-dimensional random vectors such that

$$\mathbb{P}[\xi_i = e_k] = \mathbb{P}[\xi_i = -e_k] = \frac{1}{2d}, \quad k = 1, \dots, d, \quad i \in \mathbb{N}.$$

Define $S_n = \xi_1 + \dots + \xi_n$, $n \in \mathbb{N}$, and $S_0 = 0$. The sequence $S_0, S_1, S_2, \dots$ is called the simple symmetric random walk on $\mathbb{Z}^d$. It is a Markov chain with transition probabilities

$$p_{i,i+e_1} = p_{i,i-e_1} = \dots = p_{i,i+e_d} = p_{i,i-e_d} = \frac{1}{2d}, \quad i \in \mathbb{Z}^d.$$

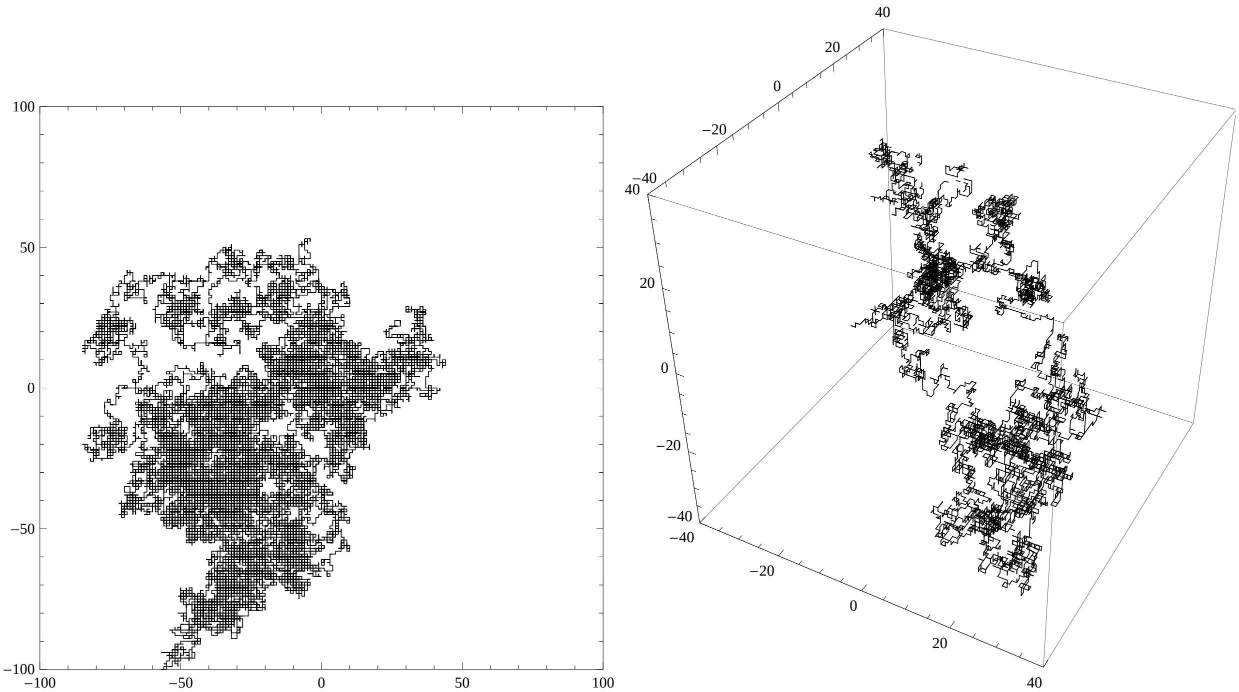

FIGURE 2. *Left:* Sample path of a simple symmetric random walk on $\mathbb{Z}^2$. *Right:* Sample path of a simple symmetric random walk on $\mathbb{Z}^3$. In both cases the random walk makes 50000 steps.

THEOREM 2.8.4 (Pólya, 1921). *The simple symmetric random walk on $\mathbb{Z}^d$ is recurrent if and only if $d = 1, 2$ and transient if and only if $d \geq 3$.*

PROOF. For $d = 1$ we already proved the statement in Theorem 2.8.2.

Consider the case $d = 2$. We compute the n-step transition probability $p_{00}^{(n)}$. For an odd n this probability is 0. For an even $n = 2k$ we have

$$p_{00}^{(2k)} = \frac{1}{4^{2k}} \sum_{i=0}^{k} \binom{2k}{i, i, k-i, k-i} = \frac{1}{4^{2k}} \binom{2k}{k} \sum_{i=0}^{k} \binom{k}{i} \binom{k}{k-i} = \left(\frac{1}{2^{2k}} \binom{2k}{k} \right)^2 \sim \frac{1}{\pi k},$$

as $k \to \infty$, where the last step is by the Stirling formula. The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges. Therefore, $\sum_{n=1}^{\infty} p_{00}^{(n)} = \infty$ and the random walk is recurrent in $d = 2$ dimensions.

Generalizing the cases $d = 1, 2$ one can show that for an arbitrary dimension $d \in \mathbb{N}$ we have, as $k \to \infty$,

$$p_{00}^{(2k)} \sim \frac{1}{(\pi k)^{d/2}}.$$

Since the series $\sum_{k=1}^{\infty} k^{-d/2}$ is convergent for $d \geq 3$ it holds that $\sum_{n=1}^{\infty} p_{00}^{(n)} < \infty$ and the random walk is transient in $d = 3$ dimensions. □

2.9. Existence and uniqueness of the invariant measure

The next two theorems state that any irreducible and recurrent Markov chain has a *unique invariant measure* λ, up to a multiplication by a constant. This measure may be finite (that is, $\sum_{i \in E} \lambda_i < +\infty$) or infinite (that is, $\sum_{i \in E} \lambda_i = +\infty$).

First we provide an explicit construction of an invariant measure for an irreducible and recurrent Markov chain. Consider a Markov chain starting at state $k \in E$. Denote the time of the first return to k by

$$T_k = \min\{n \in \mathbb{N} : X_n = k\} \in \mathbb{N} \cup \{+\infty\}.$$

The minimum of an empty set is by convention $+\infty$. For a state $i \in E$ denote the expected number of visits to i before the first return to k by

$$\gamma_i = \gamma_i^{(k)} = \mathbb{E}_k \sum_{n=0}^{T_k - 1} \mathbb{1}_{\{X_n = i\}} \in [0, +\infty].$$

Theorem 2.9.1. *For an irreducible and recurrent Markov chain starting at state $k \in E$ we have*

(1) $\gamma_k = 1$.
(2) *For all $i \in E$ it holds that $0 < \gamma_i < \infty$.*
(3) $\gamma = (\gamma_i)_{i \in E}$ *is an invariant measure.*

Proof.

Step 1. We show that $\gamma_k = 1$. By definition of T_k, we have $\sum_{n=0}^{T_k - 1} \mathbb{1}_{\{X_n = k\}} = 1$, if the chain starts at k. It follows that $\gamma_k = \mathbb{E}_k 1 = 1$.

Step 2. We show that for every state $j \in E$,

$$\gamma_j = \sum_{i \in E} p_{ij} \gamma_i. \tag{2.9.1}$$

(At this moment, both sides of (2.9.1) are allowed to be infinite, but in Step 3 we will show that both sides are actually finite). The Markov chain is recurrent, thus $T_k < \infty$ almost surely. By definition, $X_{T_k} = k = X_0$. We have

$$\gamma_j = \mathbb{E}_k \sum_{n=1}^{T_k} \mathbb{1}_{\{X_n = j\}} = \mathbb{E}_k \sum_{n=1}^{\infty} \mathbb{1}_{\{X_n = j, n \leq T_k\}} = \sum_{n=1}^{\infty} \mathbb{P}_k[X_n = j, T_k \geq n].$$

Before visiting state j at time n the chain must have been in some state i at time $n-1$, where $i \in E$ can be, in general, arbitrary. We obtain that

$$\gamma_j = \sum_{i \in E} \sum_{n=1}^{\infty} \mathbb{P}_k[X_n = j, X_{n-1} = i, T_k \geq n] = \sum_{i \in E} \sum_{n=1}^{\infty} p_{ij} \mathbb{P}_k[X_{n-1} = i, T_k \geq n].$$

Introducing the new summation variable $m = n-1$, we obtain that

$$\gamma_j = \sum_{i \in E} p_{ij} \sum_{m=0}^{\infty} \mathbb{E}_k \mathbb{1}_{\{X_m = i, T_k \geq m+1\}} = \sum_{i \in E} p_{ij} \, \mathbb{E}_k \sum_{m=0}^{T_k - 1} \mathbb{1}_{\{X_m = i\}} = \sum_{i \in E} p_{ij} \gamma_i.$$

This proves that (2.9.1) holds.

Step 3. Let $i \in E$ be an arbitrary state. We show that $0 < \gamma_i < \infty$. Since the chain is irreducible, there exist $n, m \in \mathbb{N}_0$ such that $p_{ik}^{(m)} > 0$ and $p_{ki}^{(n)} > 0$. From (2.9.1) it follows that

$$\gamma_i = \sum_{l \in E} p_{li}^{(n)} \gamma_l \geq p_{ki}^{(n)} \gamma_k = p_{ki}^{(n)} > 0.$$

On the other hand, again using (2.9.1), we obtain that

$$1 = \gamma_k = \sum_{l \in E} p_{lk}^{(m)} \gamma_l \geq p_{ik}^{(m)} \gamma_i.$$

This implies that $\gamma_i \leq 1/p_{ik}^{(m)} < \infty$. □

The next theorem states the uniqueness of the invariant measure, up to multiplication by a constant.

Theorem 2.9.2. *Consider an irreducible and recurrent Markov chain and fix some state $k \in E$. Then, every invariant measure λ can be represented in the form*

$$\lambda_j = c\gamma_j^{(k)} \quad \text{for all } j \in E,$$

where c is a constant (not depending on j). In fact, $c = \lambda_k$.

Remark 2.9.3. Hence, the invariant measure is unique up to a multiplication by a constant. In particular, the invariant measures $(\gamma_i^{(k_1)})_{i \in E}$ and $(\gamma_i^{(k_2)})_{i \in E}$, for different states $k_1, k_2 \in E$, differ by a multiplicative constant.

Proof. Let λ be an invariant measure.

Step 1. We show that $\lambda_j \geq \lambda_k \gamma_j^{(k)}$ for all $j \in E$. We will *not* use the irreducibility and the recurrence of the chain in this step. The invariance of the measure λ implies that

$$\lambda_j = \sum_{i_0 \in E} \lambda_{i_0} p_{i_0 j} = \sum_{i_0 \neq k} \lambda_{i_0} p_{i_0 j} + \lambda_k p_{kj}.$$

Applying the same procedure to λ_{i_0}, we obtain

$$\lambda_j = \sum_{i_0 \neq k} \left(\sum_{i_1 \neq k} \lambda_{i_1} p_{i_1 i_0} + \lambda_k p_{k i_0} \right) p_{i_0 j} + \lambda_k p_{kj}$$

$$= \sum_{i_0 \neq k} \sum_{i_1 \neq k} \lambda_{i_1} p_{i_1 i_0} p_{i_0 j} + \left(\lambda_k p_{kj} + \lambda_k \sum_{i_0 \neq k} p_{k i_0} p_{i_0 j} \right).$$

Applying the procedure to λ_{i_1} and repeating it over and over again we obtain that for every $n \in \mathbb{N}$,

$$\lambda_j = \sum_{i_0, i_1, \ldots, i_n \neq k} \lambda_{i_n} p_{i_n i_{n-1}} \ldots p_{i_1 i_0} p_{i_0 j} + \lambda_k \left(p_{kj} + \sum_{i_0 \neq k} p_{k i_0} p_{i_0 j} + \ldots + \sum_{i_0, \ldots, i_{n-1} \neq k} p_{k i_0} p_{i_0 i_1} \ldots p_{i_{n-1} j} \right).$$

Noting that the first term is non-negative, we obtain that

$$\lambda_j \geq 0 + \lambda_k \mathbb{P}_k[X_1 = j, T_k \geq 1] + \lambda_k \mathbb{P}_k[X_2 = j, T_k \geq 2] + \ldots + \lambda_k \mathbb{P}_k[X_n = j, T_k \geq n].$$

Since this holds for every $n \in \mathbb{N}$, we can pass to the limit as $n \to \infty$:

$$\lambda_j \geq \lambda_k \sum_{n=1}^{\infty} \mathbb{P}_k[X_n = j, T_k \geq n] = \lambda_k \gamma_j^{(k)}.$$

It follows that $\lambda_j \geq \lambda_k \gamma_j^{(k)}$.

Step 2. We prove the converse inequality. Consider $\mu_j := \lambda_j - \lambda_k \gamma_j^{(k)}$, $j \in E$. By the above, $\mu_j \geq 0$ for all $j \geq 0$ so that $\mu = (\mu_j)_{j \in E}$ is a measure. Moreover, this measure is invariant because it is a linear combination of two invariant measures. Finally, note that by definition, $\mu_k = 0$. We will prove that this implies that $\mu_j = 0$ for all $j \in E$. By the irreducibility of our Markov chain, for every $j \in E$ we can find $n \in \mathbb{N}_0$ such that $p_{jk}^{(n)} > 0$. By the invariance property of μ,

$$0 = \mu_k = \sum_{i \in E} \mu_i p_{ik}^{(n)} \geq \mu_j p_{jk}^{(n)}.$$

It follows that $\mu_j p_{jk}^{(n)} = 0$ but since $p_{jk}^{(n)} > 0$, we must have $\mu_j = 0$. By the definition of μ_j this implies that $\lambda_j = \lambda_k \gamma_j^{(k)}$. □

We can now summarize Theorems 2.9.1 and 2.9.2 as follows:

Theorem 2.9.4. *A recurrent, irreducible Markov chain has unique (up to a constant multiple) invariant measure.*

This invariant measure may be finite or infinite. However, if the Markov chain has only finitely many states, then the measure must be finite and we can even normalize it to be a *probability* measure.

Corollary 2.9.5. *A finite and irreducible Markov chain has a unique invariant probability measure.*

Proof. A finite and irreducible Markov chain is recurrent by Theorem 2.7.10. By Theorem 2.9.1, there exists an invariant measure $\lambda = (\lambda_i)_{i\in E}$. Since the number of states in E is finite by assumption and $\lambda_i < \infty$ by Theorem 2.9.1, we have $M := \sum_{i\in E} \lambda_i < \infty$ and hence, the measure λ is finite. To obtain an invariant *probability* measure, consider the measure $\lambda_i' = \lambda_i/M$.

To show that the invariant probability measure is unique, assume that we have two invariant probability measures $\nu' = (\nu_i')_{i\in E}$ and $\nu'' = (\nu_i'')_{i\in E}$. Take an arbitrary state $k \in E$. By Theorem 2.9.2, there are constants c' and c'' such that $\nu_i' = c'\gamma_i^{(k)}$ and $\nu_i'' = c''\gamma_i^{(k)}$, for all $i \in E$. But since both ν' and ν'' are *probability* measures, we have

$$1 = \sum_{i\in E} \nu_i' = c' \sum_{i\in E} \gamma_i^{(k)}, \quad 1 = \sum_{i\in E} \nu_i'' = c'' \sum_{i\in E} \gamma_i^{(k)}.$$

This implies that $c' = c''$ and hence, the measures ν' and ν'' are equal. □

Above, we considered only irreducible, recurrent chains. What happens if the chain is irreducible and transient? It turns out that in this case everything is possible:

(1) It is possible that there is no invariant measure at all (except the zero measure).
(2) It is possible that there is a unique (up to multiplication by a constant) invariant measure.
(3) It is possible that there are at least two invariant measures which are not constant multiples of each other.

Exercise 2.9.6. Consider a Markov chain on $\mathbb{N}$ with transition probabilities $p_{i,i+1} = 1$, for all $i \in \mathbb{N}$. Show that the only invariant measure is $\lambda_i = 0$, $i \in \mathbb{N}$.

Exercise 2.9.7. Consider a Markov chain on $\mathbb{Z}$ with transition probabilities $p_{i,i+1} = 1$, for all $i \in \mathbb{Z}$. Show that the invariant measures have the form $\lambda_i = c$, $i \in \mathbb{Z}$, where $c \geq 0$ is constant.

Exercise 2.9.8. Consider a simple random walk on $\mathbb{Z}$ with $p \neq \frac{1}{2}$. Show that any invariant measure has the form

$$\lambda_i = c_1 + c_2 \left(\frac{p}{1-p}\right)^i, \quad i \in \mathbb{Z},$$

for some constants $c_1 \geq 0$, $c_2 \geq 0$.

2.10. Positive recurrence and null recurrence

The set of recurrent states of a Markov chain can be further subdivided into the set of positive recurrent states and the set of negative recurrent states. Let us define the notions of positive recurrence and null recurrence.

Consider a Markov chain on state space E. Take some state $i \in E$, assume that the Markov chain starts at state i and denote by T_i the time of the first return of the chain to state i:

$$T_i = \min\{n \in \mathbb{N} : X_n = i\} \in \mathbb{N} \cup \{+\infty\}.$$

Denote by m_i the expected return time of the chain to state i, that is

$$m_i = \mathbb{E}_i T_i \in (0, \infty]$$

Note that for a transient state i we always have $m_i = +\infty$ because the random variable T_i takes the value $+\infty$ with strictly positive probability $1 - f_i > 0$, see Theorem 2.7.3. However, for a recurrent state i the value of m_i may be both finite and infinite, as we shall see later.

Definition 2.10.1. A state $i \in E$ as called *positive recurrent* if $m_i < \infty$.

Definition 2.10.2. A state $i \in E$ is called *null recurrent* if it is recurrent and $m_i = +\infty$.

Remark 2.10.3. Both null recurrent states and positive recurrent states are recurrent. For null recurrent states this is required by definition. For a positive recurrent state we have $m_i < \infty$ which means that T_i cannot attain the value $+\infty$ with strictly positive probability and hence, state i is recurrent.

Theorem 2.10.4. *Consider an irreducible Markov chain. Then the following statements are equivalent:*

(1) *Some state is positive recurrent.*
(2) *All states are positive recurrent.*
(3) *The chain has invariant* **probability** *measure* $\lambda = (\lambda_i)_{i \in E}$.

Also, if these statements hold, then $m_i = \frac{1}{\lambda_i}$ *for all* $i \in E$.

Proof. The implication $2 \Rightarrow 1$ is evident.

Proof of $1 \Rightarrow 3$. Let $k \in E$ be a positive recurrent state. Then, k is recurrent and all states of the chain are recurrent by irreducibility. By Theorem 2.9.1, $(\gamma_i^{(k)})_{i \in E}$ is an invariant measure. However, we need an invariant *probability* measure. To construct it, note that

$$\sum_{j \in E} \gamma_j^{(k)} = m_k < \infty$$

(since k is positive recurrent). We can therefore define $\lambda_i = \gamma_i^{(k)}/m_k$, $i \in E$. Then, $\sum_{i \in E} \lambda_i = 1$, and $(\lambda_i)_{i \in E}$ is an invariant probability measure.

Proof of $3 \Rightarrow 2$. Let $(\lambda_i)_{i \in E}$ be an invariant probability measure. First we show that $\lambda_k > 0$ for every state $k \in E$. Since λ is a probability measure, we have $\lambda_l > 0$ for at least one $l \in E$. By irreducibility, we have $p_{lk}^{(n)} > 0$ for some $n \in \mathbb{N}_0$ and by invariance of λ, we have

$$\lambda_k = \sum_{i \in E} p_{ik}^{(n)} \lambda_i \geq p_{lk}^{(n)} \lambda_l > 0.$$

This proves that $\lambda_k > 0$ for every $k \in E$.

By Step 1 from the proof of Theorem 2.9.2 (note that this step does not use recurrence), we have for all $j \in E$,

$$\lambda_i \geq \lambda_k \gamma_i^{(k)}.$$

Hence,

$$m_k = \sum_{i \in E} \gamma_i^{(k)} \leq \sum_{i \in E} \frac{\lambda_i}{\lambda_k} = \frac{1}{\lambda_k} < \infty.$$

It follows that k is positive recurrent, thus establishing statement 2.

PROOF THAT $m_k = \frac{1}{\lambda_k}$. Assume that statements 1,2,3 hold. In particular, the chain is recurrent and by Theorem 2.9.2, we must have $\lambda_i = \lambda_k \gamma_i^{(k)}$ for all $i \in E$. It follows that

$$m_k = \sum_{i \in E} \gamma_i^{(k)} = \sum_{i \in E} \frac{\lambda_i}{\lambda_k} = \frac{1}{\lambda_k},$$

thus proving the required formula. □

EXAMPLE 2.10.5. Any state in a *finite* irreducible Markov chain is positive recurrent. Indeed, such a chain has an invariant probability measure by Corollary 2.9.5.

EXAMPLE 2.10.6. Consider a simple symmetric random walk on $\mathbb{Z}$ or on $\mathbb{Z}^2$. This chain is irreducible. Any state is recurrent by Pólya's Theorem 2.8.4. We show that in fact, any state is *null* recurrent. To see this, note that the measure assigning the value 1 to every state $i \in E$ is invariant by the definition of the chain. By Theorem 2.9.2, any other invariant measure must be of the form $\lambda_i = c$, $i \in E$, for some constant $c \geq 0$. However, no measure of this form is a probability measure. So, there is no invariant probability measure and by Theorem 2.10.4, all states must be null recurrent.

2.11. Convergence to the invariant probability measure

We are going to state and prove a "strong law of large numbers" for Markov chains. First recall that the usual strong law of large numbers states that if $\xi_1, \xi_2, \dots$ are i.i.d. random variables with $\mathbb{E}|\xi_1| < \infty$, then

$$\frac{\xi_1 + \ldots + \xi_n}{n} \xrightarrow[n\to\infty]{a.s.} \mathbb{E}\xi_1. \tag{2.11.1}$$

The statement is not applicable if $\mathbb{E}|\xi_1| = \infty$. However, it is an exercise to show that if $\xi_1, \xi_2, \dots$ are i.i.d. random variables which are a.s. *nonnegative* with $\mathbb{E}\xi_1 = +\infty$, then

$$\frac{\xi_1 + \ldots + \xi_n}{n} \xrightarrow[n\to\infty]{a.s.} +\infty. \tag{2.11.2}$$

Consider a Markov chain $\{X_n : n \in \mathbb{N}_0\}$ with initial distribution $\alpha = (\alpha_i)_{i\in E}$. Given a state $i \in E$, denote the number of visits to state i in the first n steps by

$$V_i(n) = \sum_{k=0}^{n-1} \mathbb{1}_{\{X_k = i\}}.$$

THEOREM 2.11.1. *Consider an irreducible Markov chain $\{X_n : n \in \mathbb{N}_0\}$ with an arbitrary initial distribution $\alpha = (\alpha_i)_{i\in E}$.*

1. *If the Markov chain is transient or null recurrent, then for all $i \in E$ it holds that*

$$\frac{V_i(n)}{n} \xrightarrow[n\to\infty]{} 0 \quad a.s. \tag{2.11.3}$$

2. *If the Markov chain is positive recurrent with invariant probability measure λ, then for all $i \in E$ it holds that*

$$\frac{V_i(n)}{n} \xrightarrow[n\to\infty]{} \lambda_i \quad a.s. \tag{2.11.4}$$

PROOF. If the chain is transient, then $V_i(n)$ stays bounded as a function of n, with probability 1. This implies (2.11.3). In the sequel, let the chain be recurrent.

For simplicity, we will assume in this proof that the chain starts in state i. Denote the time of the k-th visit of the chain to i by S_k, that is

$$S_1 = \min\{n \in \mathbb{N} : X_n = i\},$$
$$S_2 = \min\{n > S_1 : X_n = i\},$$
$$S_3 = \min\{n > S_2 : X_n = i\},$$

and so on. Note that $S_1, S_2, S_3, \ldots$ are a.s. finite by the recurrence of the chain. Let also $\xi_1, \xi_2, \xi_3, \ldots$ be the excursion times between the returns to i, that is

$$\xi_1 = S_1, \quad \xi_2 = S_2 - S_1, \quad \xi_3 = S_3 - S_2, \quad \ldots .$$

Then, $\xi_1, \xi_2, \xi_3, \ldots$ are i.i.d. random variables by the Markov property.

By definition of $V_i(n)$ we have

$$\xi_1 + \xi_2 + \ldots + \xi_{V_i(n)-1} \le n \le \xi_1 + \xi_2 + \ldots + \xi_{V_i(n)}.$$

Dividing this by $V_i(n)$ we get

$$\frac{\xi_1 + \xi_2 + \ldots + \xi_{V_i(n)-1}}{V_i(n)} \le \frac{n}{V_i(n)} \le \frac{\xi_1 + \xi_2 + \ldots + \xi_{V_i(n)}}{V_i(n)}. \tag{2.11.5}$$

Note that by recurrence, $V_i(n) \underset{n\to\infty}{\longrightarrow} \infty$ a.s.

CASE 1. Let the chain be *null* recurrent. It follows that $\mathbb{E}\xi_1 = \infty$. By using (2.11.2) and (2.11.5), we obtain that

$$\frac{n}{V_i(n)} \overset{a.s.}{\underset{n\to\infty}{\longrightarrow}} \infty.$$

This proves (2.11.3).

CASE 2. Let the chain be *positive* recurrent. Then, by Theorem 2.10.4, $\mathbb{E}\xi_1 = m_i = \frac{1}{\lambda_i} < \infty$. Using (2.11.1) and (2.11.5) we obtain that

$$\frac{n}{V_i(n)} \overset{a.s.}{\underset{n\to\infty}{\longrightarrow}} \frac{1}{\lambda_i}.$$

This proves (2.11.4). □

In the next theorem we prove that the n-step transition probabilities converge, as $n \to \infty$, to the invariant probability measure.

THEOREM 2.11.2. *Consider an irreducible, aperiodic, positive recurrent Markov chain $\{X_n : n \in \mathbb{N}_0\}$ with transition matrix P and invariant probability measure $\lambda = (\lambda_i)_{i\in E}$. The initial distribution $\alpha = (\alpha_i)_{i\in E}$ may be arbitrary. Then, for all $j \in E$ it holds that*

$$\lim_{n\to\infty} \mathbb{P}[X_n = j] = \lambda_j.$$

In particular, $\lim_{n\to\infty} p_{ij}^{(n)} = \lambda_j$ for all $i, j \in E$.

REMARK 2.11.3. In particular, the theorem applies to any irreducible and aperiodic Markov chain with finite state space.

For the proof we need the following lemma.

Lemma 2.11.4. *Consider an irreducible and aperiodic Markov chain. Then, for every states* $i, j \in E$ *we can find* $N = N(i, j) \in \mathbb{N}$ *such that for all* $n > N$ *we have* $p_{ij}^{(n)} > 0$.

Proof. The chain is irreducible, hence we can find $r \in \mathbb{N}_0$ such that $p_{ij}^{(r)} > 0$. Also, the chain is aperiodic, hence we can find $N_0 \in \mathbb{N}$ such that for all $k > N_0$ we have $p_{ii}^{(k)} > 0$. It follows that for all $k > N_0$,

$$p_{ij}^{(k+r)} > p_{ii}^{(k)} p_{ij}^{(r)} > 0.$$

It follows that for every $n := k + r$ such that $n > N_0 + r$, we have $p_{ij}^{(n)} > 0$. □

Proof of Theorem 2.11.2. We use the "coupling method".

Step 1. Consider *two* Markov chains called $\{X_n : n \in \mathbb{N}_0\}$ and $\{Y_n : n \in \mathbb{N}_0\}$ such that

(1) X_n is a Markov chain with initial distribution α and transition matrix P.
(2) Y_n is a Markov chain with initial distribution λ (the invariant probability measure) and the same transition matrix P.
(3) The process $\{X_n : n \in \mathbb{N}_0\}$ is independent of the process $\{Y_n : n \in \mathbb{N}_0\}$.

Note that both Markov chains have the same transition matrix but different initial distributions. Fix an arbitrary state $b \in E$. Denote by T be the time at which the chains meet at state b:

$$T = \min\{n \in \mathbb{N} : X_n = Y_n = b\} \in \mathbb{N} \cup \{+\infty\}.$$

If the chains do not meet at b, we set $T = +\infty$.

Step 2. We show that $\mathbb{P}[T < \infty] = 1$. Consider the stochastic process $W_n = (X_n, Y_n)$ taking values in $E \times E$. It is a Markov chain on $E \times E$ with transition probabilities given by

$$\tilde{p}_{(i,k),(j,l)} = p_{ij} p_{kl}, \quad (i, k) \in E \times E, \quad (j, l) \in E \times E.$$

The initial distribution of W_0 is given by

$$\mu_{(i,k)} = \alpha_i \lambda_k, \quad (i, k) \in E \times E.$$

Since the chains X_n and Y_n are aperiodic and irreducible by assumption of the theorem, we can apply Lemma 2.11.4 to obtain for every $i, j, k, l \in E$ a number $N = N(i, j, k, l) \in \mathbb{N}$ such that for all $n > N$ we have

$$\tilde{p}_{(i,k),(j,e)}^{(n)} = p_{ij}^{(n)} p_{ke}^{(n)} > 0.$$

Thus, the chain W_n is irreducible. Also, it is an exercise to check that the probability measure $\tilde{\lambda}_{(i,k)} := \lambda_i \lambda_k$ is invariant for W_n. Thus, by Theorem 2.10.4, the Markov chain W_n is positive recurrent and thereby recurrent. Therefore, $T < \infty$ a.s. by Lemma 2.7.11.

Step 3. Define the stochastic process $\{Z_n : n \in \mathbb{N}_0\}$ by

$$Z_n = \begin{cases} X_n, & \text{if } n \le T, \\ Y_n, & \text{if } n \ge T. \end{cases}$$

Then, Z_n is a Markov chain with initial distribution α and the same transition matrix P as X_n and Y_n. (The Markov chain Z_n is called the coupling of X_n and Y_n). The chain Y_n starts with the invariant probability measure λ and hence, at every time n, Y_n is distributed according to λ. Also, the chain Z_n has the same initial distribution α and the same transition matrix P as the chain X_n, so that in particular, the random elements X_n and Z_n have the same distribution at every time n. Using these facts, we obtain that

$$|\mathbb{P}[X_n = j] - \lambda_j| = |\mathbb{P}[X_n = j] - \mathbb{P}[Y_n = j]| = |\mathbb{P}[Z_n = j] - \mathbb{P}[Y_n = j]|.$$

By definition of Z_n, we can rewrite this as

$$\begin{aligned} |\mathbb{P}[X_n = j] - \lambda_j| &= |\mathbb{P}[X_n = j, n < T] + \mathbb{P}[Y_n = j, n \geq T] - \mathbb{P}[Y_n = j]| \\ &= |\mathbb{P}[X_n = j, n < T] - \mathbb{P}[Y_n = j, n < T]| \\ &\leq \mathbb{P}[T > n]. \end{aligned}$$

But we have shown in Step 2 that $\mathbb{P}[T = \infty] = 0$, hence $\lim_{n\to\infty} \mathbb{P}[T > n] = 0$. It follows that

$$\lim_{n\to\infty} \mathbb{P}[X_n = j] = \lambda_j,$$

thus establishing the theorem. □