CHAPTER 3

Renewal processes and Poisson process

3.1. Definition of renewal processes and limit theorems

Let $\xi_1, \xi_2, \ldots$ be independent and identically distributed random variables with $\mathbb{P}[\xi_k > 0] = 1$. Define their partial sums

$$S_n = \xi_1 + \ldots + \xi_n, \quad n \in \mathbb{N}, \quad S_0 = 0.$$

Note that the sequence $S_1, S_2, \ldots$ is increasing. We call $S_1, S_2, \ldots$ the *renewal times* (or simply *renewals*) and $\xi_1, \xi_2, \ldots$ the *interrenewal times*.

Definition 3.1.1. The process $\{N_t : t \geq 0\}$ given by

$$N_t = \sum_{n=1}^{\infty} \mathbb{1}_{\{S_n \leq t\}}$$

is called the *renewal process*.

Theorem 3.1.2 (Law of large numbers for renewal processes). *Let $m := \mathbb{E}\xi_1 \in (0, \infty)$, then*

$$\frac{N_t}{t} \overset{a.s.}{\to} \frac{1}{m}, \quad \textit{as } t \to \infty.$$

Idea of proof. By the definition of N_t we have the inequality

$$S_{N_t} \leq t \leq S_{N_t+1}.$$

Dividing this by N_t we obtain

$$\frac{S_{N_t}}{N_t} \leq \frac{t}{N_t} \leq \frac{S_{N_t+1}}{N_t+1} \cdot \frac{N_t+1}{N_t}. \tag{3.1.1}$$

We have $N_t \to \infty$ as $t \to \infty$ since there are infinitely many renewals and thus, the function N_t (which is non-decreasing by definition) cannot stay bounded. By the law of large numbers, both sides of (3.1.1) a.s. converge to m as $t \to \infty$. By the sandwich lemma, we have

$$\frac{t}{N_t} \overset{a.s.}{\to} m, \quad \text{as } t \to \infty.$$

This proves the claim. □

Theorem 3.1.3 (Central limit theorem for renewal processes). *Let $m := \mathbb{E}\xi_1 \in (0, \infty)$ and $\sigma^2 := \operatorname{Var} \xi_1 \in (0, \infty)$. Then,*

$$\frac{N_t - \frac{t}{m}}{\frac{\sigma}{m^{3/2}}\sqrt{t}} \overset{d}{\to} \mathrm{N}(0,1), \quad \textit{as } t \to \infty.$$

Idea of proof. The usual central limit theorem for $S_n = \xi_1 + \ldots + \xi_n$ states that

$$\frac{S_n - nm}{\sigma\sqrt{n}} \xrightarrow[n\to\infty]{d} \mathrm{N}(0,1).$$

Denoting by N a standard normal random variable we can write this as follows: For large n, we have an approximate equality of distributions

$$S_n \approx nm + \sigma\sqrt{n}N.$$

This means that the interval $[0, nm + \sigma\sqrt{n}N]$ contains approximately n renewals. By the law of large numbers for renewal processes, see Theorem 3.1.2, it seems plausible that the interval $[nm, nm + \sigma\sqrt{n}N]$ contains approximately $\sigma\sqrt{n}N/m$ renewals. It follows that the interval $[0, nm]$ contains approximately $n - \sigma\sqrt{n}N/m$ renewals. Let us now introduce the variable $t = nm$. Then, $n \to \infty$ is equivalent to $t \to \infty$. Consequently, for large t in the interval $[0, t]$ we have approximately

$$\frac{t}{m} - \frac{\sigma\sqrt{t}}{m^{3/2}}N$$

renewals. By definition, this number of renewals is N_t. This means that

$$\frac{N_t - \frac{t}{m}}{\frac{\sigma}{m^{3/2}}\sqrt{t}} \approx N,$$

for large t. □

Definition 3.1.4. The *renewal function* $H(t)$ is the expected number of renewals in the interval $[0, t]$:

$$H(t) = \mathbb{E}N_t, \quad t \geq 0.$$

Remark 3.1.5. Denoting by $F^{*k}(t) = \mathbb{P}[S_k \leq t]$ the distribution function of S_k, we have the formula

$$H(t) = \mathbb{E}N_t = \mathbb{E}\sum_{k=1}^{\infty} \mathbb{1}_{S_k \leq t} = \sum_{k=1}^{\infty} \mathbb{E}\mathbb{1}_{S_k \leq t} = \sum_{k=1}^{\infty} \mathbb{P}[S_k \leq t] = \sum_{k=1}^{\infty} F^{*k}(t).$$

Theorem 3.1.6 (Weak renewal theorem). *Let $m := \mathbb{E}\xi_1 \in (0, \infty)$. It holds that*

$$\lim_{t\to\infty} \frac{H(t)}{t} = \frac{1}{m}.$$

Idea of proof. By Theorem 3.1.2, $\frac{N_t}{t} \overset{a.s.}{\to} \frac{1}{m}$ as $t \to \infty$. In order to obtain Theorem 3.1.6, we have to take expectation of both sides and interchange the limit and the expectation. The rigorous justification will be omitted. □

Definition 3.1.7. The random variables ξ_k are called *lattice* if there are $a > 0, b \in \mathbb{R}$ so that ξ_k with probability 1 takes values in the set $a\mathbb{Z} + b$, that is

$$\mathbb{P}[\xi_k \in \{an + b : n \in \mathbb{Z}\}] = 1.$$

Theorem 3.1.8 (Blackwell renewal theorem). *Assume that ξ_1 is non-lattice and let $m := \mathbb{E}\xi_1 \in (0, \infty)$. Then, for all $s > 0$,*

$$\lim_{t\to\infty} (H(t+s) - H(t)) = \frac{s}{m}.$$

Proof. Omitted □

3.2. Stationary processes and processes with stationary increments

Consider a stochastic process $\{X_t, t \geq 0\}$. For concreteness, we have chosen the index set T to be $[0, \infty)$, but similar definitions apply to stochastic processes with index sets $T = \mathbb{R}, \mathbb{N}, \mathbb{N}_0, \mathbb{Z}$.

Definition 3.2.1. The process $\{X_t : t \geq 0\}$ is called *stationary* if for all $n \in \mathbb{N}$, $0 \leq t_1 \leq \ldots \leq t_n$ and all $h \geq 0$,

$$(X_{t_1}, \ldots, X_{t_n}) \stackrel{d}{=} (X_{t_1+h}, \ldots, X_{t_n+h}).$$

Example 3.2.2. Let $\{X_t : t \in \mathbb{N}_0\}$ be independent and identically distributed random variables. We claim that the process X is stationary. Let μ be the probability distribution of X_t, that is $\mu(A) = \mathbb{P}[X_t \in A]$, for all Borel sets $A \subset \mathbb{R}$. Then, for all Borel sets $A_1, \ldots, A_n \subset \mathbb{R}$,

$$\mathbb{P}[X_{t_1+h} \in A_1, \ldots, X_{t_n+h} \in A_n] = \mu(A_1) \cdot \ldots \cdot \mu(A_n) = \mathbb{P}[X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n].$$

This proves that X is stationary.

Example 3.2.3. Let $\{X_t : t \in \mathbb{N}_0\}$ be a Markov chain starting with an *invariant* probability distribution λ. Then, X_t is stationary.

Proof. Let us first compute the joint distribution of $(X_h, X_{h+1}, \ldots, X_{h+m})$. For any states $i_0, \ldots, i_m \in E$ we have

$$\mathbb{P}[X_h = i_0, X_{h+1} = i_1, \ldots, X_{h+m} = i_m] = \mathbb{P}[X_h = i_0] \cdot p_{i_0 i_1} \cdot \ldots \cdot p_{i_{m-1} i_m}.$$

Since the initial measure λ of the Markov chain is invariant, we have $\mathbb{P}[X_h = i_0] = \lambda_{i_0}$. We therefore obtain that

$$\mathbb{P}[X_h = i_0, X_{h+1} = i_1, \ldots, X_{h+m} = i_m] = \lambda_{i_0} p_{i_0 i_1} \cdot \ldots \cdot p_{i_{m-1} i_m}.$$

This expression does not depend on h thus showing that

$$(X_h, X_{h+1}, \ldots, X_{h+m}) \stackrel{d}{=} (X_0, X_1, \ldots, X_m).$$

If we drop some components in the first vector and the corresponding components in the second vector, the vectors formed by the remaining components still have the same distribution. In this way we can prove that $(X_{t_1+h}, X_{t_2+h}, \ldots, X_{t_n+h})$ has the same distribution as $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$. □

Definition 3.2.4. The process $\{X_t : t \geq 0\}$ has *stationary increments* if for all $n \in \mathbb{N}$, $h \geq 0$ and $0 \leq t_0 \leq t_1 \leq \ldots \leq t_m$, we have the following equality in distribution:

$$(X_{t_1+h} - X_{t_0+h}, X_{t_2+h} - X_{t_1+h}, \ldots, X_{t_n+h} - X_{t_{n-1}+h}) \stackrel{d}{=} (X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}).$$

Definition 3.2.5. The process $\{X_t : t \geq 0\}$ has *independent increments* if for all $n \in \mathbb{N}$ and $0 \leq t_0 \leq t_1 \leq \ldots \leq t_n$, the random variables

$$X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$$

are independent.

Later we will consider two examples of processes which have both stationary and independent increments: the Poisson Process and the Brownian Motion.

3.3. Poisson process

The Poisson process is a special case of renewal process in which the interrenewal times are exponentially distributed. Namely, let $\xi_1, \xi_2, \ldots$ be independent identically distributed random variables having exponential distribution with parameter $\lambda > 0$, that is

$$\mathbb{P}[\xi_k \leq x] = 1 - e^{-\lambda x}, \quad x \geq 0.$$

Define the renewal times S_n by

$$S_n = \xi_1 + \ldots + \xi_n, \quad n \in \mathbb{N}, \quad S_0 = 0.$$

It's an exercise to show (for example, by induction) that the density of S_n is given by

$$f_{S_n}(x) = \frac{\lambda^n x^{n-1}}{(n-1)!} e^{-\lambda x}, \quad x \geq 0.$$

The distribution of S_n is called the Erlang distribution with parameters n and λ. It is a particular case of the Gamma distribution.

DEFINITION 3.3.1. The Poisson process with intensity $\lambda > 0$ is a process $\{N_t : t \geq 0\}$ defined by

$$N_t = \sum_{k=1}^{\infty} \mathbb{1}_{\{S_k \leq t\}}.$$

Note that N_t counts the number of renewals in the interval $[0, t]$. The next theorem explains why the Poisson process was named after Poisson.

THEOREM 3.3.2. *For all $t \geq 0$ it holds that $N_t \sim \operatorname{Poi}(\lambda t)$.*

PROOF. We need to prove that for all $n \in \mathbb{N}_0$,

$$\mathbb{P}[N_t = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

STEP 1. Let first $n = 0$. Then,

$$\mathbb{P}[N_t = 0] = \mathbb{P}[\xi_1 > t] = e^{-\lambda t},$$

thus establishing the required formula for $n = 0$.

STEP 2. Let $n \in \mathbb{N}$. We compute the probability $\mathbb{P}[N_t = n]$. By definition of N_t we have

$$\mathbb{P}[N_t = n] = \mathbb{P}[N_t \geq n] - \mathbb{P}[N_t \geq n+1] = \mathbb{P}[S_n \leq t] - \mathbb{P}[S_{n+1} \leq t].$$

Using the formula for the density of S_n we obtain that

$$\mathbb{P}[N_t = n] = \int_0^t f_{S_n}(x)dx - \int_0^t f_{S_{n+1}}(x)dx = \int_0^t \left(\frac{\lambda^n x^{n-1}}{(n-1)!} e^{-\lambda x} - \frac{\lambda^{n+1} x^n}{n!} e^{-\lambda x} \right) dx.$$

The expression under the sign of the integral is equal to

$$\frac{d}{dx} \left(\frac{(\lambda x)^n}{n!} e^{-\lambda x} \right).$$

Thus, we can compute the integral as follows:

$$\mathbb{P}[N_t = n] = \left(\frac{(\lambda x)^n}{n!} e^{-\lambda x} \right) \Bigg|_{x=0}^{x=t} = \frac{(\lambda t)^n}{n!} e^{-\lambda t},$$

where the last step holds since we assumed that $n \neq 0$. $\square$

REMARK 3.3.3. From the above theorem it follows that the renewal function of the Poisson process is given by $H(t) = \mathbb{E}N_t = \lambda t$.

For the next theorem let $U_1, \ldots, U_n$ be independent random variables which are uniformly distributed on the interval $[0, t]$. Denote by $U_{(1)} \leq \ldots \leq U_{(n)}$ the order statistics of $U_1, \ldots, U_n$.

THEOREM 3.3.4. *The conditional distribution of the random vector* $(S_1, \ldots, S_n)$ *given that* $\{N_t = n\}$ *coincides with the distribution of* $(U_{(1)}, \ldots, U_{(n)})$*:*

$$(S_1, \ldots, S_n)|\{N_t = n\} \stackrel{d}{=} (U_{(1)}, \ldots, U_{(n)}).$$

PROOF. We will compute the densities of both vectors and show these densities are equal.

STEP 1. The joint density of the random variables $(\xi_1, \ldots, \xi_{n+1})$ has (by independence) the product form

$$f_{\xi_1,\ldots,\xi_{n+1}}(u_1, \ldots, u_{n+1}) = \prod_{k=1}^{n+1} \lambda e^{-\lambda u_k}, \quad u_1, \ldots, u_{n+1} > 0.$$

STEP 2. We compute the joint density of $(S_1, \ldots, S_{n+1})$. Consider a linear transformation A defined by

$$A(u_1, u_2, \ldots, u_{n+1}) = (u_1, u_1 + u_2, \ldots, u_1 + \ldots + u_{n+1}).$$

The random variables $(S_1, \ldots, S_{n+1})$ can be obtained by applying the linear transformation A to the variables $(\xi_1, \ldots, \xi_{n+1})$:

$$(S_1, \ldots, S_{n+1}) = A(\xi_1, \ldots, \xi_{n+1}).$$

The determinant of the transformation A is 1 since the matrix of this transformation is triangular with 1's on the diagonal. By the density transformation theorem, the density of $(S_1, \ldots, S_{n+1})$ is given by

$$f_{S_1,\ldots,S_{n+1}}(t_1, \ldots, t_{n+1}) = \prod_{k=1}^{n+1} \lambda e^{-\lambda(t_k - t_{k-1})} = \lambda^{n+1} e^{-\lambda t_{n+1}},$$

where $0 = t_0 < t_1 < \ldots < t_{n+1}$. Otherwise, the density vanishes. Note that the formula for the density depends only on t_{n+1} and does not depend on $t_1, \ldots, t_n$.

STEP 3. We compute the conditional density of $(S_1, \ldots, S_n)$ given that $N_t = n$. Let $0 < t_1 < \ldots < t_n < t$. Intuitively, the conditional density of $(S_1, \ldots, S_n)$ given that $N_t = n$ is given by

$$\begin{aligned}
f_{S_1,\ldots,S_n}(t_1, \ldots, t_n | N_t = n) &= \lim_{\varepsilon \downarrow 0} \frac{\mathbb{P}[t_1 < S_1 < t_1 + \varepsilon, \ldots, t_n < S_1 < t_n + \varepsilon | N_t = n]}{\varepsilon^n} \\
&= \lim_{\varepsilon \downarrow 0} \frac{\mathbb{P}[t_1 < S_1 < t_1 + \varepsilon, \ldots, t_n < S_n < t_n + \varepsilon, N_t = n]}{\varepsilon^n \mathbb{P}[N_t = n]} \\
&= \lim_{\varepsilon \downarrow 0} \frac{\mathbb{P}[t_1 < S_1 < t_1 + \varepsilon, \ldots, t_n < S_n < t_n + \varepsilon, S_{n+1} > t]}{\varepsilon^n \mathbb{P}[N_t = n]}.
\end{aligned}$$

Using the formula for the joint density of $(S_1, \dots, S_{n+1})$ and noting that this density does not depend on $t_1, \dots, t_n$, we obtain that

$$\frac{\mathbb{P}[t_1 < S_1 < t_1 + \varepsilon, \dots, t_n < S_n < t_n + \varepsilon, S_{n+1} > t]}{\varepsilon^n \mathbb{P}[N_t = n]} = \frac{\int_t^\infty \lambda^{n+1} e^{-\lambda t_{n+1}} dt_{n+1}}{\mathbb{P}[N_t = n]} = \frac{n!}{t^n},$$

where in the last step we used that N_t has Poisson distribution with parameter λt. So, we have

$$f_{S_1,\dots,S_n}(t_1, \dots, t_n | N_t = n) = \begin{cases} \frac{n!}{t^n}, & \text{for } 0 < t_1 < \dots < t_n < t, \\ 0, & \text{otherwise.} \end{cases}$$

Step 4. The joint density of the order statistics $(U_{(1)}, \dots, U_{(n)})$ is known (Stochastik I) to be given by

$$f_{U_{(1)},\dots,U_{(n)}}(t_1, \dots, t_n) = \begin{cases} \frac{n!}{t^n}, & \text{for } 0 < t_1 < \dots < t_n < t, \\ 0, & \text{otherwise.} \end{cases}$$

This coincides with the conditional density of $(S_1, \dots, S_n)$ given that $N_t = n$, thus proving the theorem. □

Theorem 3.3.5. *The Poisson process $\{N_t : t \geq 0\}$ has independent increments and these increments have Poisson distribution, namely for all $t, s \geq 0$ we have*

$$N_{t+s} - N_t \sim \text{Poi}(\lambda s).$$

Proof. Take some points $0 = t_0 \leq t_1 \leq \dots \leq t_n$. We determine the distribution of the random vector

$$(N_{t_1}, N_{t_2} - N_{t_1}, \dots, N_{t_n} - N_{t_{n-1}}).$$

Take some $x_1, \dots, x_n \in \mathbb{N}_0$. We compute the probability

$$P := \mathbb{P}[N_{t_1} = x_1, N_{t_2} - N_{t_1} = x_2, \dots, N_{t_n} - N_{t_{n-1}} = x_n].$$

Let $x = x_1 + \dots + x_n$. By definition of conditional probability,

$$P = \mathbb{P}[N_{t_1} = x_1, N_{t_2} - N_{t_1} = x_2, \dots, N_{t_n} - N_{t_{n-1}} = x_n | N_{t_n} = x] \cdot \mathbb{P}[N_{t_n} = x].$$

Given that $N_{t_n} = x$, the Poisson process has x renewals in the interval $[0, t_n]$ and by Theorem 3.3.4 these renewals have the same distribution as x independent random variables which have uniform distribution on the interval $[0, t_n]$, after arranging them in an increasing order. Hence, in order to compute the conditional probability we can use the multinomial distribution:

$$P = \left(\frac{x!}{x_1! \dots x_n!} \prod_{k=1}^n \frac{(t_k - t_{k-1})^{x_k}}{t_n^{x_k}} \right) \cdot \frac{(\lambda t_n)^x}{x!} e^{-\lambda t_n}.$$

After making transformations we arrive at

$$P = \prod_{k=1}^n \left(\frac{(\lambda(t_k - t_{k-1}))^{x_k}}{x_k!} e^{-\lambda(t_k - t_{k-1})} \right).$$

From this formula we see that the random variables N_{t_1}, $N_{t_2} - N_{t_1}, \dots,$ $N_{t_n} - N_{t_{n-1}}$ are independent and that they are Poisson distributed, namely

$$N_{t_k} - N_{t_{k-1}} \sim \text{Poi}(\lambda(t_k - t_{k-1})).$$

This proves the theorem. □

Theorem 3.3.6. *The Poisson process has stationary increments.*

Proof. Take some $h \geq 0$, and some $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$. We have to show that the distribution of the random vector

$$(N_{t_1+h} - N_{t_0+h}, N_{t_2+h} - N_{t_1+h}, \ldots, N_{t_n+h} - N_{t_{n-1}+h})$$

does not depend on h. However, we know from Theorem 3.3.5 that the components of this vector are independent and that

$$N_{t_k+h} - N_{t_{k-1}+h} \sim \mathrm{Poi}(\lambda(t_k - t_{k-1})),$$

which does not depend on h. □

3.4. Lattice renewal processes

In this section we show how the theory of Markov chains can be used to obtain some properties of renewal processes whose interrenewal times are integer. Let $\xi_1, \xi_2, \ldots$ be independent and identically distributed random variables with values in $\mathbb{N} = \{1, 2, \ldots\}$. Let us write

$$r_n := \mathbb{P}[\xi_1 = n], \quad n \in \mathbb{N}.$$

We will make the aperiodicity assumption:

$$\gcd\{n \in \mathbb{N} : r_n \neq 0\} = 1. \tag{3.4.1}$$

For example, this condition excludes renewal processes for which the ξ_k's take only even values. Define the renewal times $S_n = \xi_1 + \ldots + \xi_n$, $n \in \mathbb{N}$.

Theorem 3.4.1. *Let $m := \mathbb{E}\xi_1$ be finite. Then,*

$$\lim_{n\to\infty} \mathbb{P}[\exists k \in \mathbb{N} : S_k = n] = \frac{1}{m}.$$

So, the probability that there is a renewal at time n converges, as $n \to \infty$, to $\frac{1}{m}$.

Proof. Step 1. Consider a Markov chain defined as follows: Let

$$X_n = \inf\{t \geq n : t \text{ is renewal time}\} - n.$$

The random variable X_n (which is called the forward renewal time) represents the length of the time interval between n and the first renewal following n. (Please think why X_n has the Markov property). Note that at renewal times we have $X_n = 0$.
The state space of this chain is

$$E = \{0, 1, \ldots, M-1\}, \text{if } M < \infty,$$
$$E = \{0, 1, 2, \ldots\}, \text{if } M = \infty,$$

where M is the maximal value which the ξ_k's can attain:

$$M = \sup\{i \in \mathbb{N} : r_i > 0\} \in \mathbb{N} \cup \{\infty\}.$$

The transition probabilities of this Markov chain are given by

$$p_{i,i-1} = 1 \text{ for } i = 1, 2, \ldots, M-1,$$
$$p_{0,i} = r_{i+1} \text{ for } i = 1, \ldots, M-1.$$

STEP 2. We prove that the chain is irreducible. Starting at any state $i \in E$ we can reach state 0 by following the path

$$i \to i-1 \to i-2 \to \ldots \to 0.$$

So, every state leads to state 0. Let us prove that conversely, state 0 leads to every state. Let first M be finite. Starting in state 0 we can reach any state $i \in E$ with positive probability by following the path

$$0 \to M-1 \to M-2 \to \ldots \to i.$$

If M is infinite, then for every $i \in E$ we can find some $K > i$ such that $r_K > 0$. Starting at state 0 we can reach state i by following the path

$$0 \to K-1 \to K-2 \to \ldots \to i.$$

We have shown that every state leads to 0 and 0 leads to every state, so the chain is irreducible.

STEP 3. We prove that the chain is aperiodic. By irreducibility, we need to show that state 0 is aperiodic. For every i such that $r_i \neq 0$ we can go from 0 to 0 in i steps by following the path

$$0 \to i-1 \to i-2 \to \ldots \to 0.$$

By (3.4.1) the greatest common divisor of all such i's is 1, so the period of state 0 is 1 and it is aperiodic.

STEP 4. We claim that the unique invariant probability measure of this Markov chain is given by

$$\lambda_i = \frac{r_{i+1} + r_{i+2} + \ldots}{m}, \quad i \in E.$$

Indeed, the equations for the invariant probability measure look as follows:

$$\lambda_j = \sum_{i=0}^{M-1} p_{ij}\lambda_i = p_{0,j}\lambda_0 + p_{j+1,j}\lambda_{j+1} = r_{j+1}\lambda_0 + \lambda_{j+1}.$$

It follows that

$$\lambda_j - \lambda_{j+1} = r_{j+1}\lambda_0.$$

We obtain the following equations:

$$\begin{aligned} \lambda_0 - \lambda_1 &= r_1\lambda_0, \\ \lambda_1 - \lambda_2 &= r_2\lambda_0, \\ \lambda_2 - \lambda_3 &= r_3\lambda_0, \\ &\ldots \end{aligned}$$

By adding all these equations starting with the $(j+1)$-st one, we obtain that

$$\lambda_j = (r_{j+1} + r_{j+2} + \ldots)\lambda_0.$$

It remains to compute λ_0. By adding the equations for all $j = 0, 1, \ldots, M-1$ we obtain that

$$1 = \lambda_0 + \lambda_1 + \ldots = (r_1 + 2r_2 + 3r_3 + \ldots)\lambda_0 = m\lambda_0.$$

It follows that

$$\lambda_0 = \frac{1}{m}.$$

This proves the formula for the invariant probability distribution.

STEP 5. Our chain is thus irreducible, aperiodic, and positive recurrent. By the theorem on the convergence to the invariant probability distribution we have

$$\lim_{n\to\infty} \mathbb{P}[X_n = 0] = \lambda_0 = \frac{1}{m}.$$

Recalling that we have $X_n = 0$ if and only if n is a renewal time, we obtain that

$$\lim_{n\to\infty} \mathbb{P}[\exists k \in \mathbb{N} : S_n = k] = \lim_{n\to\infty} \mathbb{P}[X_n = 0] = \frac{1}{m},$$

thus proving the claim of the theorem. $\square$