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Winter term 2013/14
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Stochastics II

Exercise Sheet 10

Due: January 8th, 2014

Note: Please submit exercise sheets in groups of two persons!

Problem 1 (6 points)

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ be a filtered probability space and assume that the filtration is right continuous. Let $T_1, T_2, \ldots$ be a sequence of stopping times defined on this space.

(a) Show that if $T_n(\omega) \downarrow T(\omega)$ $\forall \omega \in \Omega$, then T is a stopping time.

(b) Show that if $T_n(\omega) \uparrow T(\omega)$ $\forall \omega \in \Omega$, then T is a stopping time.

Problem 2 (6 points)

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ be a filtered probability space and let T be a stopping time defined on this space. Define a family of sets $\mathcal{F}_T = \{A \in \mathcal{F} : \forall t \geq 0 \;\; A \cap \{T \leq t\} \in \mathcal{F}_t\}$.

(a) Show that this is a σ-algebra.

(b) Show that if S and T are two stopping times, then $\mathcal{F}_S \cap \mathcal{F}_T = \mathcal{F}_{\min(S,T)}$.

(c) Show that if $S(\omega) \leq T(\omega)$ for all $\omega \in \Omega$, then $\mathcal{F}_S \subseteq \mathcal{F}_T$.

Problem 3 (6 points)

Let $\{B(s) : s \geq 0\}$ be a standard Brownian motion. Take $a > 0$ and define $T_a = \inf\{s \geq 0 : B(s) = a\}$.

(a) Show that for $u < v < a$ and $t > 0$,

$$\mathbb{P}[T_a < t, u < B(t) < v] = \mathbb{P}[2a - v \leq B(t) \leq 2a - u].$$

(b) Define $M(t) = \sup\limits_{0\leq s\leq t} B(s)$. Determine the joint distribution function of the random vector $(M(t), B(t))$.

Problem 4 (6 points)

Let $B = \{B(t) : t \geq 0\}$ be a Brownian motion. Show that the process $\{W(t) : t \geq 0\}$ given by

$$W(t) = \begin{cases} tB(\frac{1}{t}), & \text{if } t > 0, \\ 0, & \text{if } t = 0 \end{cases}$$

is again a Brownian motion.

Hint: Don't forget to show that the process W is continuous, especially at $t = 0$. In order to show that $\lim_{s\to\infty} B(s)/s = 0$ a.s. one can first argue that this limit relation holds if s is restricted to $\mathbb{N}$ and then define $D_n = \sup_{0\leq u\leq 1}(B(n + u) - B(n))$ and show that $\lim_{n\to\infty} D_n/n = 0$.