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Winter term 2013/14
January 8th, 2014

Stochastics II

Exercise Sheet 11

Due: January 15th, 2014

Note: Please submit exercise sheets in groups of two persons!

Problem 1 (6 points)

Let $X : \Omega \to \mathbb{R}$ be a square integrable random variable on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{A} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. The conditional variance $\operatorname{Var}(X\,|\mathcal{A})$ of X given $\mathcal{A}$ is defined by $\operatorname{Var}(X\,|\mathcal{A}) = \mathbb{E}(X^2\,|\mathcal{A}) - (\mathbb{E}(X\,|\mathcal{A}))^2$. Show that

$$\operatorname{Var}(X) = \mathbb{E}(\operatorname{Var}(X\,|\mathcal{A})) + \operatorname{Var}(\mathbb{E}(X\,|\mathcal{A})).$$

Problem 2 (6 points)

Let $X_1, X_2, \dots$ be a sequence of independent random variables with $\mathbb{E}X_i = 0$ and $\operatorname{Var}(X_i) = \sigma_i^2 < \infty$, for all $i \in \mathbb{N}$. Define

$$S_n = \sum_{i=1}^n X_i, \quad V_n = \operatorname{Var} S_n = \sum_{i=1}^n \sigma_i^2, \quad S_0 = V_0 = 0.$$

Show that $\{S_n^2 - V_n : n \in \mathbb{N}_0\}$ is a martingale w.r.t. the natural filtration generated by the random variables $X_1, X_2, \dots$.

Problem 3 (6 points)

Let (B_1, B_2) be a 2-dimensional standard Brownian motion, that is $\{B_1(t) : t \geq 0\}$ and $\{B_2(t) : t \geq 0\}$ are independent one-dimensional standard Brownian motions. Consider a stopping time $T_a = \inf\{t \geq 0 : \; B_1(t) = a\}$, for $a > 0$. Define $V_a = B_2(T_a)$. Show that the density of V_a is given by

$$f_{V_a}(t) = \frac{1}{\pi} \frac{a}{a^2 + t^2}, \quad t \in \mathbb{R},$$

i.e. V_a is a Cauchy-distributed random variable.

Problem 4 (6 points)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{A} \subset \mathcal{F}$ a sub-σ-algebra of $\mathcal{F}$. Let furthermore $X : \Omega \to \mathbb{R}$ be a square integrable random variable, that is $\mathbb{E}[X^2] < \infty$.

(a) Show that for all $\mathcal{A}$-measurable random variables $Z : \Omega \to \mathbb{R}$ with $\mathbb{E}[Z^2] < \infty$ it holds that

$$\mathbb{E}\left[(X - \mathbb{E}(X\,|\mathcal{A}) + Z)^2\right] = \mathbb{E}\left[(X - \mathbb{E}(X\,|\mathcal{A}))^2\right] + \mathbb{E}[Z^2].$$

(b) Show that for all $\mathcal{A}$-measurable random variables $Y : \Omega \to \mathbb{R}$ with $\mathbb{E}[Y^2] < \infty$ it holds that

$$\mathbb{E}\left[(X - Y)^2\right] \geq \mathbb{E}\left[(X - \mathbb{E}(X\,|\mathcal{A}))^2\right].$$

Remark. Part (b) means that among all random variables depending only on the information contained in the σ-algebra $\mathcal{A}$, the best approximation to X (in the mean square sense) is the conditional expectation $\mathbb{E}[X|\mathcal{A}]$.