Problem 1 (6 points)
Let \( X : \Omega \rightarrow \mathbb{R} \) be a square integrable random variable on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and let \( \mathcal{A} \subset \mathcal{F} \) be a sub-\(\sigma\)-algebra of \( \mathcal{F} \). The conditional variance \( \text{Var}(X | \mathcal{A}) \) of \( X \) given \( \mathcal{A} \) is defined by \( \text{Var}(X | \mathcal{A}) = \mathbb{E}(X^2 | \mathcal{A}) - \left( \mathbb{E}(X | \mathcal{A}) \right)^2 \). Show that
\[
\text{Var}(X) = \mathbb{E}(\text{Var}(X | \mathcal{A})) + \text{Var}(\mathbb{E}(X | \mathcal{A})).
\]

Problem 2 (6 points)
Let \( X_1, X_2, \ldots \) be a sequence of independent random variables with \( \mathbb{E}X_i = 0 \) and \( \text{Var}(X_i) = \sigma_i^2 < \infty \), for all \( i \in \mathbb{N} \). Define
\[
S_n = \sum_{i=1}^{n} X_i, \quad V_n = \text{Var} S_n = \sum_{i=1}^{n} \sigma_i^2, \quad S_0 = V_0 = 0.
\]
Show that \( \{S_n^2 - V_n : n \in \mathbb{N}_0\} \) is a martingale w.r.t. the natural filtration generated by the random variables \( X_1, X_2, \ldots \).

Problem 3 (6 points)
Let \((B_1, B_2)\) be a 2-dimensional standard Brownian motion, that is \( \{B_1(t) : t \geq 0\} \) and \( \{B_2(t) : t \geq 0\} \) are independent one-dimensional standard Brownian motions. Consider a stopping time \( T_a = \inf\{t \geq 0 : B_1(t) = a\} \), for \( a > 0 \). Define \( V_a = B_2(T_a) \). Show that the density of \( V_a \) is given by
\[
f_{V_a}(t) = \frac{1}{\pi a^2 + t^2}, \quad t \in \mathbb{R},
\]
i.e. \( V_a \) is a Cauchy-distributed random variable.

Problem 4 (6 points)
Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \( \mathcal{A} \subset \mathcal{F} \) a sub-\(\sigma\)-algebra of \( \mathcal{F} \). Let furthermore \( X : \Omega \rightarrow \mathbb{R} \) be a square integrable random variable, that is \( \mathbb{E}[X^2] < \infty \).

(a) Show that for all \( \mathcal{A} \)-measurable random variables \( Z : \Omega \rightarrow \mathbb{R} \) with \( \mathbb{E}[Z^2] < \infty \) it holds that
\[
\mathbb{E}\left[ (X - \mathbb{E}(X | \mathcal{A}) + Z)^2 \right] = \mathbb{E}\left[ (X - \mathbb{E}(X | \mathcal{A}))^2 \right] + \mathbb{E}[Z^2].
\]

(b) Show that for all \( \mathcal{A} \)-measurable random variables \( Y : \Omega \rightarrow \mathbb{R} \) with \( \mathbb{E}[Y^2] < \infty \) it holds that
\[
\mathbb{E}\left[ (X - Y)^2 \right] \geq \mathbb{E}\left[ (X - \mathbb{E}(X | \mathcal{A}))^2 \right].
\]

Remark. Part (b) means that among all random variables depending only on the information contained in the \(\sigma\)-algebra \( \mathcal{A} \), the best approximation to \( X \) (in the mean square sense) is the conditional expectation \( \mathbb{E}[X | \mathcal{A}] \).