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Winter term 2013/14
January 15th, 2014

Stochastics II

Exercise Sheet 12

Due: January 22nd, 2014

Note: Please submit exercise sheets in groups of two persons!

Problem 1 (3 + 3 points)

Let $\xi_1, \xi_2, \ldots$ be a sequence of i.i.d. random variables with

$$\mathbb{P}[\xi_k = 1] = \mathbb{P}[\xi_k = -1] = p \in (0, 1/2) \text{ and } \mathbb{P}[\xi_k = 0] = 1 - 2p.$$

Define $S_n = \xi_1 + \cdots + \xi_n$, $S_0 = 0$ and $T = \min\{n \in \mathbb{N} : \ S_n \in \{-a, b\}\}$ with $a, b \in \mathbb{N}$.

(a) Show that $S_n^2 - 2pn$ is a martingale.

(b) Calculate $\mathbb{E}T$.

Hint: In order to show that T is finite a.s. you can estimate $\mathbb{P}[T > n] \leq \mathbb{P}[-a < S_n < b]$ by the central limit theorem.

Problem 2 (2 + 2 + 2 + 2 + 1 points)

Let $\{X_n : n \in \mathbb{N}_0\}$ and $\{Y_n : n \in \mathbb{N}_0\}$ be submartingales w.r.t. the filtration $\{\mathcal{F}_n\}_{n \in \mathbb{N}_0}$ and $m, n \in \mathbb{N}$. Prove the following statements:

(a) $\mathbb{E}\left[X_n \,|\mathcal{F}_m\right] \geq X_m$, for all $m < n$.

(b) $\mathbb{E}\left[X_n\right] \geq \mathbb{E}\left[X_m\right]$, for all $m < n$.

(c) $\{X_n \vee Y_n : n \in \mathbb{N}_0\}$ is a submartingale w.r.t. $\{\mathcal{F}_n\}_{n \in \mathbb{N}_0}$. Here, $x \vee y$ denotes $\max(x, y)$.

(d) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an *increasing* and *convex* function such that $\mathbb{E}|\varphi(X_n)| < \infty$ for all $n \in \mathbb{N}$. Then $\{\varphi(X_n) : n \in \mathbb{N}_0\}$ is a submartingale w.r.t. $\{\mathcal{F}_n\}_{n \in \mathbb{N}_0}$.

(e) Give an example of a submartingale $\{X_n : n \in \mathbb{N}_0\}$, such that $\{X_n^2 : n \in \mathbb{N}_0\}$ is not a submartingale. *Hint:* X_n does not have to be random.

Problem 3 (3 points)

Let X and Y be random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}Y^2 = \mathbb{E}X^2 < \infty$ and let $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra. Show that if $\mathbb{E}[Y|\mathcal{G}] = X$ a.s. then it holds $X = Y$ a.s.

Problem 4 (6 points)

Let $\xi_1, \xi_2, \ldots$ be a sequence of independent random variables with $\mathbb{E}\xi_i = 0$ for all $i \in \mathbb{N}$. Show that for each $k \in \mathbb{N}$ the sequence $\{X_n^{(k)} : n \in \mathbb{N}_0\}$ given by

$$X_n^{(k)} = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \xi_{i_1} \ldots \xi_{i_k}, \quad X_0 = 0,$$

is a martingale w.r.t. the filtration generated by $\xi_1, \xi_2, \ldots$.