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Winter term 2013/14
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Stochastics II

Exercise Sheet 13

Due: January 29nd, 2014

Note: Please submit exercise sheets in groups of two persons!

Problem 1 (6 points)

Let $\xi_1, \ldots, \xi_n$ be a sequence of i.i.d. random variables with $\mathbb{E}|\xi_1| < \infty$ and define $S_n = \sum_{k=1}^n \xi_k$. Calculate $\mathbb{E}[\xi_n | \sigma(S_n)]$, where $\sigma(S_n)$ denotes the σ-algebra generated by S_n.

Problem 2 (2 + 2 + 2 points)

Let $\{B(t) : t \geq 0\}$ be a standard Brownian motion. Show that the following stochastic processes are martingales w.r.t. the natural filtration.

(a) $\{B^2(t) - t : t \geq 0\}$.

(b) $\{e^{uB(t) - t\frac{u^2}{2}} : t \geq 0\}$, for every fixed $u \in \mathbb{R}$.

(c) $\{B^3(t) - 3tB(t) : t \geq 0\}$.

Problem 3 (3 + 3 points)

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with

$$\mathbb{P}[X_i = 1] = p_+, \quad \mathbb{P}[X_i = -1] = p_-, \quad \mathbb{P}[X_i = 0] = q, \quad i \in \mathbb{N},$$

where $p_+, p_-, q \in (0, 1)$ such that $p_+ + p_- + q = 1$ and $p_+ \neq p_-$. Define $S_n = X_1 + \ldots + X_n$, $S_0 = 0$.

(a) Show that there exists some $\alpha \neq 0$, such that $\{e^{\alpha S_n} : n \in \mathbb{N}_0\}$ is a martingale w.r.t. the natural filtration.

(b) Define $T_y = \min\{n \in \mathbb{N} : S_n = y\}$. Compute $\mathbb{P}[T_b < T_a]$, for $a < 0 < b$, $a, b \in \mathbb{Z}$.

Problem 4 (6 points)

Show that if a family C of random variables is bounded in L^p, for some $p > 1$ (meaning that there is a constant B such that $\mathbb{E}|X|^p < B$ for all $X \in C$ and B does not depend on the choice of X), then this family is uniformly integrable.

Problem 5 (voluntary)

A function $f : \mathbb{Z}^2 \to \mathbb{R}$ is called harmonic, if

$$f(x, y) = \frac{1}{4}\left(f(x, y+1) + f(x, y-1) + f(x-1, y) + f(x+1, y)\right)$$

for all $(x, y) \in \mathbb{Z}^2$.

(a) Show that any bounded harmonic function is constant.

(b) Show that any non-negative harmonic function is constant.

Hint: Consider the simple symmetric random walk $X_0, X_1, X_2, \ldots$ on $\mathbb{Z}^2$. If the function f is harmonic, then $f(X_n)$ is a martingale.