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Stochastics II

Exercise Sheet 14

Due: February 5th, 2014

Note: Please submit exercise sheets in groups of two persons!

Problem 1 (6 points)

Let $g : [0, \infty) \to (0, \infty)$ be a monotone increasing function with

$$\lim_{x\to+\infty} \frac{g(x)}{x} = +\infty.$$

Show that if a sequence of random variables $X_1, X_2, \dots$ satisfies $\sup_{n\in\mathbb{N}} \mathbb{E}[g(|X_n|)] < \infty$, then it is uniformly integrable.

Problem 2 (4 + 2 points)

Let $\{B(t) \colon t \ge 0\}$ be a standard Brownian motion. For $a > 0$ denote by

$$T = \inf\{t \ge 0 : |B(t)| = a\}$$

the first exit time of B from the strip $(-a, a)$.

(a) Compute $\mathbb{E}e^{-\lambda T}$ for $\lambda > 0$.

(b) Compute $\mathbb{E}[T^2]$. (For example, you may wish to take the derivative of the Laplace transform in λ).

Problem 3 (4 + 2 points)

Let $\{B(t) \colon t \ge 0\}$ be a standard Brownian motion. For $\mu > 0$ (the drift) and $a > 0$ (the level) denote by

$$T = \inf\{t \ge 0 \colon B(t) - \mu t = a\} \in (0, +\infty]$$

the first time the drifted Brownian motion $B(t) - \mu t$ hits the level a. (If the level a is never reached, then $T = +\infty$).

(a) By using the martingale $e^{\theta B(t) - \frac{1}{2}\theta^2 t}$ with a suitable $\theta > 0$ show that

$$\mathbb{E}e^{-\lambda T} = e^{-a\left(\mu + \sqrt{\mu^2+2\lambda}\right)}, \quad \lambda > 0.$$

(b) By letting $\lambda \downarrow 0$ compute the ruin probability $\mathbb{P}[T \ne +\infty]$.

Problem 4 (6 points)

Show that a sequence of random variables $X_1, X_2, \dots$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is uniformly integrable if and only if

(a) $\sup_{n\in\mathbb{N}} \mathbb{E}|X_n| < \infty$ and

(b) for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $n \in \mathbb{N}$ and all sets $A \in \mathcal{F}$ satisfying $\mathbb{P}[A] < \delta$ it holds that

$$\mathbb{E}[|X_n| \mathbb{1}_A] < \varepsilon.$$