Junior-Prof. Dr. Z. Kabluchko
Stefan Roth, Judith Schmidt

Winter term 2013/14
February 6th, 2014

Stochastics II

Exercise Sheet 15

Please don't submit solutions

Problem 1

Consider a Markov chain on the state space $E = \{1, 2, 3, 4\}$ with transition matrix

$$P = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 2/3 \\ 1 & 0 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 \end{pmatrix}.$$

(a) Identify the communicating classes of this Markov chain.

(b) Which of these classes are closed? Which classes are recurrent?

Problem 2

At all times, an urn contains N balls – some white balls and some black balls. At each stage, a coin having probability p, $0 < p < 1$, of landing heads is flipped. If heads appears, then a ball is drawn at random from the urn and is replaced by a white ball; if tails appears, then a ball is drawn at random from the urn and is replaced by a black ball. Let X_n denote the number of white balls in the urn after the n-th stage. Then $X_0, X_1, \ldots$ forms a Markov chain.

(a) Compute the transition probabilities and determine the communicating classes of this chain.

(b) Compute $\lim_{n\to\infty} \mathbb{P}(X_n = k)$ for $k = 0, 1, \ldots, N$.

Problem 3

On a 8×8 chessboard a bishop (der Läufer) moves according to the usual chess rules. It starts in the left top corner of the chessboard and at any moment of time performs a move chosen at random from the set of all moves allowed by the chess rules. All moves allowed by the rules are equiprobable. What is (approximately) the probability that after 10^6 moves the bishop returns to the left top corner of the chessboard?

Problem 4

Let $\{B(t) : t \geq 0\}$ be a standard Brownian motion. Compute

(a) $\mathbb{P}[B(1) < B(2) < B(3)]$.

(b) $\mathbb{P}[B(1) < B(3) < B(2)]$.

Problem 5

Let $\{N(t) : t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$. Compute

$$\mathbb{P}[N(1) = 1, N(2) = 2, N(3) = 3].$$

Problem 6

Let $\{B(t) : t \geq 0\}$ be a standard Brownian motion. Define the set $Z := \{t \geq 0 : \ B(t) = 0\}$. Show that

$$\mathbb{P}[\mathrm{Leb}(Z) = 0] = 1,$$

where Leb denotes the Lebesgue measure on $\mathbb{R}$.

Problem 7

Assume that the positions of the stars are modeled by a Poisson process on $\mathbb{R}^3$ with intensity equal to the Lebesgue measure. Let R be the distance from the origin to the closest star. Compute the distribution function of R.

Problem 8

Prove that the Brownian motion and the Poisson process are stochastically continuous processes.

Problem 9

Let X be a square integrable random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{A} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. Show that $\mathbb{E}(X|\mathcal{A})$ is the orthogonal projection of X onto $L_2(\Omega, \mathcal{A}, \mathbb{P})$, meaning that

$$\mathbb{E}[Y(X - \mathbb{E}(X|\mathcal{A}))] = 0$$

for every square integrable, $\mathcal{A}$-measurable random variable Y on Ω.

Problem 10

Let $\{N(t) : t \geq 0\}$ be a renewal process with F being the distribution function of the corresponding interarrival times. Let F^{*n} be the n-th convolution power of F. Show that for all $t \geq 0$,

$$\mathbb{E}N^2(t) = \sum_{n=1}^{\infty}(2n-1)F^{*n}(t).$$

Problem 11

Let $X_1, X_2, \ldots$ be absolutely continuous i.i.d. random variables with density function f_0 (which is strictly positive everywhere) and let f_1 be another density function. For $n \in \mathbb{N}$ define

$$L_n = \prod_{k=1}^{n} \frac{f_1(X_k)}{f_0(X_k)}, \quad L_0 = 1.$$

Show that $\{L_n : n \in \mathbb{N}_0\}$ is a martingale w.r.t. the natural filtration $\mathcal{F}_n = \sigma\{X_1, \ldots, X_n\}$.

Problem 12

Let $\{X_n : n \in \mathbb{N}_0\}$ and $\{Y_n : n \in \mathbb{N}_0\}$ be two martingales w.r.t. some filtration $\{\mathcal{F}_n\}_{n \in \mathbb{N}_0}$. Show that if $X_0 = Y_0 = 0$, then

$$\mathbb{E}[X_n Y_n] = \sum_{k=1}^{n} \mathbb{E}[(X_k - X_{k-1})(Y_k - Y_{k-1})].$$

Problem 13

Let $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\{|X_n|^p : n \in \mathbb{N}\}$, $\{|Y_n|^q : n \in \mathbb{N}_0\}$ be uniformly integrable families of random variables. Show that the family $\{X_n Y_n\}_{n\in\mathbb{N}}$ is uniformly integrable as well.

Problem 14

Let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables with $\mathbb{P}[\xi_k > 0] = 1$ and $\mathbb{E}\xi_k = 1$. Show that

$$X_n := \xi_1 \cdot \ldots \cdot \xi_n, \quad n \in \mathbb{N}, \quad X_0 = 1,$$

is a martingale.

Problem 15

Let $N, \xi_1, \xi_2, \ldots$ be independent random variables such that $N \sim \mathrm{Poi}(\lambda)$ and $\mathbb{P}[\xi_k = 1] = p$, $\mathbb{P}[\xi_k = 0] = 1 - p$. Compute the distribution of the random variable

$$S := \xi_1 + \ldots + \xi_N.$$

Problem 16

Let $\xi_1, \xi_2, \ldots$ be independent random variables with $\mathbb{P}[\xi_k \le t] = 1 - e^{-t}$, $t \ge 0$. Show that

$$\limsup_{n\to\infty} \frac{X_n}{\log n} = 1 \quad \text{a.s.}$$