Problem 1 (6 points)

Let \( \{X_t : t \in [0,1]\} \) be a stochastic process with continuous sample paths defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Show that the following subsets of \( \Omega \) are measurable:

(a) \( \{\omega \in \Omega : \sup_{t \in [0,1]} X_t(\omega) > 1\} \).

(b) \( \{\omega \in \Omega : \exists t \in [0,1] \text{ such that } X_t(\omega) = 0\} \).

(c) \( \{\omega \in \Omega : \text{the function } t \mapsto X_t(\omega) \text{ is increasing}\} \).

Problem 2 (6 points)

Consider a stochastic process \( \{X_t : t \in [0,1]\} \) consisting of independent, identically distributed random variables \( X_t \) such that \( X_t \) is Bernoulli with parameter \( 1/2 \), for every \( t \in [0,1] \). Show that this process is not stochastically continuous and not continuous in \( L^1 \).

Problem 3 (6 points)

Construct two stochastic processes \( \{X_t : t \in [0,1]\} \) and \( \{Y_t : t \in [0,1]\} \) whose one-dimensional distributions coincide but such that the two-dimensional distributions of these processes are different. (That is, the random variable \( X_t \) should have the same distribution as the random variable \( Y_t \) for every \( t \in [0,1] \), but there should exist \( t_1, t_2 \in [0,1] \) for which the distributions of the random vectors \( (X_{t_1}, X_{t_2}) \) and \( (Y_{t_1}, Y_{t_2}) \) are different).

Problem 4 (6 points)

Let \( \mathbb{R}^N \) be the set of all functions \( f : \mathbb{N} \to \mathbb{R} \). Show that the following subsets of \( \mathbb{R}^N \) belong to the cylinder \( \sigma \)-algebra:

(a) The set of all functions \( f : \mathbb{N} \to \mathbb{R} \) such that \( \lim_{n \to \infty} f(n) = 0 \).

(b) The set of all functions \( f : \mathbb{N} \to \mathbb{R} \) which are increasing, that is \( f(n+1) > f(n) \) for every \( n \).

(c) The set of all functions \( f : \mathbb{N} \to \mathbb{R} \) such that \( \lim_{n \to \infty} f(n) \) exists.

Hint: If \( B \) is a Borel subset of \( \mathbb{R}^n \) and \( t_1, \ldots, t_n \in \mathbb{N} \), then the set \( \{f : \mathbb{N} \to \mathbb{R} : (f(t_1), \ldots, f(t_n)) \in B\} \) is in the cylinder \( \sigma \)-algebra. You can use this without proof.