Junior-Prof. Dr. Z. Kabluchko
Stefan Roth, Judith Schmidt

Winter term 2013/14
November 13th, 2013

Stochastik II

Exercise Sheet 5

Due: November 20th, 2013

Note: Please submit exercise sheets in groups of two persons!

Problem 1 (6 points)

A fair die is rolled repeatedly. Denote by X_n the number of different numbers observed in the rolls $1, \ldots, n$. For example, if the rolls are $3, 5, 3, 1, 4, 6, 6, 2, \ldots$, then $X_1 = 1$, $X_2 = 2$, $X_3 = 2$, $X_4 = 3$, $X_5 = 4$, $X_6 = 5$, $X_7 = 5$ and $X_k = 6$ for $k \geq 8$.

(a) Write down the transition matrix of the Markov chain X_n.

(b) Let $N = \min\{n \in \mathbb{N} : X_n = 6\}$ be the first time at which all 6 numbers have been observed. Compute $\mathbb{E}N$, the expectation of N.

Hint: If the chain is in state 5, what is the expected time needed to reach state 6?

Problem 2 (6 points)

On a 8×8 chessboard a rook (der Turm) moves according to the usual chess rules. At any moment of time the rook performs a move chosen at random from the set of all moves allowed by the chess rules. All moves allowed by the rules are equiprobable. Let us agree that the rook is not allowed to stay in the same field. What is (approximately) the probability that after 10^6 moves the rook is located in the left top corner of the chessboard?

Problem 3 (6 points)

A fly performs a random walk on the vertices of the cube. At each step it remains where it is with probability $1/4$, or moves to one of its neighbouring vertices each having probability $1/4$. Let A and B be two diametrally opposite vertices of the cube.

(a) If the walk starts at A, what is the mean number of steps until it returns to A?

(b) If the walk starts at A, what is the expected number of visits to B until its first return to A?

Problem 4 (6 points)

Consider a Markov chain on the state space $E = \{1, 2, \ldots\}$ with the following transition probabilities. If a Markov chain is in state $i \in E$, then it moves to state $i+1$ with probability $i/(i+1)$ or falls down to state 1 with probability $1/(i+1)$. The chain starts at state 1. Let $T = \min\{n \in \mathbb{N} : X_n = 1\}$ be its first return time to state 1.

(a) Compute $\mathbb{P}[T > k]$, $\mathbb{P}[T = k]$ (in particular, $\mathbb{P}[T = +\infty]$) and $\mathbb{E}T$.

(b) Ist this chain irreducible? Ist this chain recurrent? Is this chain positive recurrent?

(c) Compute all invariant measures of this chain. By invariant measures we mean not necessarily probability measures.