Junior-Prof. Dr. Z. Kabluchko
Stefan Roth, Judith Schmidt

Winter term 2013/14
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Stochastics II

Exercise Sheet 6

Due: November 27, 2013

Note: Please submit exercise sheets in groups of two persons!

Problem 1 (4 points)

Let $\{N_t : t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$. Find:

(a) $\mathbb{P}[N_1 = 2, N_2 = 3, N_3 = 5]$.

(b) $\mathbb{P}[N_1 \leq 2, N_2 = 3, N_3 \geq 5]$.

Hint: You can use that the Poisson process has independent increments.

Problem 2 (6 points)

Show that the Poisson process $\{N_t : \ t \geq 0\}$ is stochastically continuous and continuous in L^2.

Problem 3 (6 points)

Let $\{N_t : \ t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$. Calculate

$$\mathbb{P}(N_s = k | N_t = n)$$

for $0 < s < t$, $n \in \mathbb{N}$ and $k = 0, 1, \ldots, n$.

Hint: As a function of k, the result is the probability mass function of some known distribution.

Problem 4 (10 points)

Let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables with $\xi_k > 0$ a.s. Define $S_k = \xi_1 + \ldots + \xi_k$, $k \in \mathbb{N}$, $S_0 = 0$, and let $\{N_t, \ t \geq 0\}$ be the renewal process defined by $N_t = \sum_{k=1}^{\infty} \mathbb{1}_{S_k \leq t}$. Consider the *excess time* (or forward renewal time) $T(t) = S_{N_t+1} - t$, the *current life time* (or backward renewal time) $C(t) = t - S_{N_t}$, and the *total life time* $D(t) = T(t) + C(t)$, where $t > 0$.

Now let $\{N : t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$.

(a) Determine the distribution of $T(t)$. (That is, compute $\mathbb{P}[T(t) \leq x]$ for $x > 0$).

(b) Show that the distribution of the current life time $C(t)$ is given by

$$\mathbb{P}(C(t) \leq s) = \begin{cases} 1 - e^{-\lambda s}, & \text{if } s < t, \\ 1, & \text{if } s = t. \end{cases}$$

(c) Determine $\mathbb{E}D(t)$.

(d) Consider the following two arguments to determine $\mathbb{E}D(t)$:

First argument: $\mathbb{E}D(t) = \mathbb{E}(S_{N_t+1} - S_{N_t}) = \mathbb{E}\xi_{N_t+1} = \frac{1}{\lambda}$.

Second argument: $\mathbb{E}D(t) = \mathbb{E}(S_{N_t+1} - S_{N_t}) = 2\mathbb{E}(S_{N_t+1} - t) = 2\mathbb{E}T(t) = \frac{2}{\lambda}$.

Both arguments give incorrect results. Where are the errors in these arguments? (Please provide an explanation why the corresponding step in the argument is wrong).