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Stochastics II

Exercise Sheet 7

Due: December 4th, 2013

Note: Please submit exercise sheets in groups of two persons!

Problem 1 (6 points)

Let $\{N^{(1)}(t) : t \geq 0\}$ and $\{N^{(2)}(t) : t \geq 0\}$ be independent Poisson processes with intensities $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively. Show, that $\{N_t\} = \{N_t^{(1)} + N_t^{(2)}\}$ is a Poisson process with intensity $\lambda_1 + \lambda_2$.

Hint: You can use the following definition of the Poisson process: $\{N_t : t \geq 0\}$ is a Poisson process with intensity $\lambda > 0$ if $N(0) = 0$, N has independent increments, and $N(t) - N(s) \sim \mathrm{Poi}(\lambda(t-s))$ for every $0 \leq s < t$.

Problem 2 (6 points)

Let $\{B^{(1)}(t), t \geq 0\}$ and $\{B^{(2)}(t) : t \geq 0\}$ be two independent standard Brownian motions. For which numbers $a_1, a_2 \in \mathbb{R}$ is $\{Y(t) : t \geq 0\}$ with $Y(t) = a_1 B^{(1)}(t) + a_2 B^{(2)}(t)$ a standard Brownian motion?

Problem 3 (6 points)

Let $\{N_t : t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$. Let Y be a random variable with $\mathbb{P}[Y = +1] = \mathbb{P}[Y = -1] = \frac{1}{2}$ which is independent of the process $\{N_t : t \geq 0\}$. Define a stochastic process $\{X_t : t \geq 0\}$ by $X_t = Y \cdot (-1)^{N_t}$.

(a) Let $t > 0$ be arbitrary but fixed. Calculate the probability that N_t is even respectively odd.

(b) Let $t > 0$ be arbitrary but fixed. Calculate the probability that X_t is 1 respectively -1.

(c) Calculate the covariance of X_t and X_s, where $s, t > 0$.

Hint: $e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!}$.

Problem 4 (6 points)

Let $X = (X_1, \ldots, X_m)^T$ be a m-dimensional standard Gaussian random vector, that is $X_1, \ldots, X_m \sim \mathrm{N}(0,1)$ and $X_1, \ldots, X_m$ are independent. Let A be any $d \times m$-matrix and $b \in \mathbb{R}^d$. Define the random vector

$$Y = AX + b.$$

(a) Prove that $\mathbb{E}Y = b$ and $\mathrm{Cov}(Y) = AA^T$. (Here, $\mathrm{Cov}(Y)$ is the covariance matrix of Y).

(b) Prove the formula for the characteristic function of Y:

$$\mathbb{E}e^{i\langle t, Y\rangle} = e^{i\langle t, b\rangle - \frac{1}{2}\langle t, \Sigma t\rangle}, \quad t \in \mathbb{R}^d,$$

where $\Sigma = AA^T$ and $\langle t, s\rangle = \sum_{k=1}^{d} t_k s_k$ is the scalar product of $t = (t_1, \ldots, t_d)$ and $s = (s_1, \ldots, s_d)$.