Stochastics II
Exercise Sheet 7
Due: December 4th, 2013

Note: Please submit exercise sheets in groups of two persons!

Problem 1 (6 points)
Let \( \{N^{(1)}(t) : t \geq 0\} \) and \( \{N^{(2)}(t) : t \geq 0\} \) be independent Poisson processes with intensities \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \), respectively. Show, that \( \{N_t = N^{(1)}_t + N^{(2)}_t\} \) is a Poisson process with intensity \( \lambda_1 + \lambda_2 \).

Hint: You can use the following definition of the Poisson process: \( \{N_t : t \geq 0\} \) is a Poisson process with intensity \( \lambda > 0 \) if \( N_0 = 0 \), \( N \) has independent increments, and \( N(t) - N(s) \sim \text{Poi}(\lambda(t-s)) \) for every \( 0 \leq s < t \).

Problem 2 (6 points)
Let \( \{B^{(1)}(t), t \geq 0\} \) and \( \{B^{(2)}(t) : t \geq 0\} \) be two independent standard Brownian motions. For which numbers \( a_1, a_2 \in \mathbb{R} \) is \( \{Y(t) : t \geq 0\} \) with \( Y(t) = a_1 B^{(1)}(t) + a_2 B^{(2)}(t) \) a standard Brownian motion?

Problem 3 (6 points)
Let \( \{N_t : t \geq 0\} \) be a Poisson process with intensity \( \lambda > 0 \). Let \( Y \) be a random variable with \( \mathbb{P}[Y = +1] = \mathbb{P}[Y = -1] = \frac{1}{2} \) which is independent of the process \( \{N_t : t \geq 0\} \). Define a stochastic process \( \{X_t : t \geq 0\} \) by \( X_t = Y \cdot (-1)^{N_t} \).

(a) Let \( t > 0 \) be arbitrary but fixed. Calculate the probability that \( N_t \) is even respectively odd.

(b) Let \( t > 0 \) be arbitrary but fixed. Calculate the probability that \( X_t \) is 1 respectively \(-1\).

(c) Calculate the covariance of \( X_t \) and \( X_s \), where \( s, t > 0 \).

Hint: \( e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!} \).

Problem 4 (6 points)
Let \( X = (X_1, \ldots, X_m)^T \) be a \( m \)-dimensional standard Gaussian random vector, that is \( X_1, \ldots, X_m \sim N(0, 1) \) and \( X_1, \ldots, X_m \) are independent. Let \( A \) be any \( d \times m \)-matrix and \( b \in \mathbb{R}^d \). Define the random vector \( Y = AX + b \).

(a) Prove that \( \mathbb{E} Y = b \) and \( \text{Cov}(Y) = AA^T \). (Here, \( \text{Cov}(Y) \) is the covariance matrix of \( Y \)).

(b) Prove the formula for the characteristic function of \( Y \):
\[
\mathbb{E} e^{i\langle t,Y \rangle} = e^{i\langle t,b \rangle - \frac{1}{2} \langle t, \Sigma t \rangle}, \quad t \in \mathbb{R}^d,
\]
where \( \Sigma = AA^T \) and \( \langle t, s \rangle = \sum_{k=1}^{d} t_k s_k \) is the scalar product of \( t = (t_1, \ldots, t_d) \) and \( s = (s_1, \ldots, s_d) \).