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Winter term 2013/14
December 4th, 2013

Stochastics II

Exercise Sheet 8

Due: December 11th, 2013

Note: Please submit exercise sheets in groups of two persons!

Problem 1 (6 points)

Let $X = \{X(t) : \ t \geq 0\}$ be a compound Poisson process, that is

$$X(t) = \sum_{i=1}^{N(t)} U_i \ , \qquad t \geq 0$$

where $N = \{N(t) : t \geq 0\}$ is a Poisson process with intensity $\lambda > 0$ and $\{U_n\}_{n\in\mathbb{N}}$ is a sequence of i.i.d. random variables which are independent of N. Define $\varphi_U(z) = \mathbb{E}\exp(iU_1 z)$, $z \in \mathbb{R}$.

(a) Show that the characteristic function of $X(t)$ is given by

$$\varphi_{X(t)}(z) = \exp(\lambda t(\varphi_U(z) - 1)).$$

(b) Derive a formula for the expected value of $X(t), t \geq 0$.

Problem 2 (6 points)

Let $\{B(t) : \ t \geq 0\}$ be a Brownian motion and let $c > 0$. Show that $\{\frac{B(ct)}{\sqrt{c}} : \ t \geq 0\}$ is again a Brownian motion.

Problem 3 (6 points)

Let $\{B(t) : \ t \geq 0\}$ be a Brownian motion. Consider the process $X(t) = \frac{B(e^t)}{e^{t/2}}$, where $t \in \mathbb{R}$.

(a) Calculate the covariance of $X(t)$ and $X(s)$, for $s, t \in \mathbb{R}$.

(b) Show that the process X is stationary.

Remark: The process X is called the Ornstein-Uhlenbeck process.

Problem 4 (6 points)

Let $X_1, X_2, \ldots$ be independent and identically distributed random vectors with values in $\mathbb{R}^d$. Let C be the covariance matrix of X_1. Show that

$$\frac{X_1 + \ldots + X_n - n\mathbb{E}X_1}{\sqrt{n}} \xrightarrow{d} N, \text{ as } n \to \infty,$$

where N is a d-dimensional Gaussian vector with mean 0 and covariance matrix C.
Hint: A sequence of d-dimensional random vectors $Y_1, Y_2, \ldots$ converges in distribution to a d-dimensional random vector Y if and only if $\varphi_{Y_n}(t) = \mathbb{E}\exp(i\langle Y_n, t\rangle)$ converges to $\varphi_Y(t) = \mathbb{E}\exp(i\langle Y, t\rangle)$ for every $t \in \mathbb{R}^d$, as $n \to \infty$.