Junior-Prof. Dr. Z. Kabluchko | Winter term 2013/14
Stefan Roth, Judith Schmidt | December 11th, 2013

# Stochastics II

## Exercise Sheet 9

Due: December 18th, 2013

Note: Please submit exercise sheets in groups of two persons!

**Problem 1** (6 points)

Let $\{B(t): t \geq 0\}$ be a standard Brownian motion. Calculate the covariance of $B^2(s)$ and $B^2(t)$, for $s, t \geq 0$.

**Problem 2** (6 points)

Let $\{B(t): t \geq 0\}$ be a standard Brownian motion. Show that

$$\sum_{k=0}^{n-1} \left( B\left(\frac{k+1}{n}\right) - B\left(\frac{k}{n}\right) \right)^2 \xrightarrow[n\to\infty]{P} 1.$$

**Problem 3** (6 points)

Let $\{B_1(t) : t \geq 0\}, \ldots, \{B_d(t) : t \geq 0\}$, where $d \in \mathbb{N}$, be $d$ independent standard Brownian motions. A $d$-dimensional Brownian motion is the $\mathbb{R}^d$-valued stochastic process $B = (B_1, \ldots, B_d)$. Let $U$ be an orthogonal $d \times d$-matrix. Show that the process $\{U \cdot B(t) : t \geq 0\}$ is also a $d$-dimensional Brownian motion.

*Remark.* A stochastic process with values in $\mathbb{R}^d$ is a collection $\{X_t : t \in T\}$, where each $X_t : \Omega \to \mathbb{R}^d$ is a random vector with values in $\mathbb{R}^d$.

**Problem 4** (6 points)

Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with $X_i \sim \mathrm{N}(0, 1)$. Show that

$$\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\log n}} = 1 \text{ a.s.}$$

*Hint:* You can use without proof that for $x > 0$ it holds that

$$\frac{1}{x + \frac{1}{x}} e^{-x^2/2} \leq \mathbb{P}[X_i \geq x] \leq \frac{1}{x} e^{-x^2/2}.$$