Analysis on Hanoi-type fractal quantum graphs

P. Alonso Ruiz, D. Kelleher, A. Teplyaev

University of Siegen and University of Connecticut

Workshop "Probability, Analysis and Geometry"
Ulm, 2 Sept. 2013

Probability, Analysis and Geometry

Einstein Relation

$$d_w X = \frac{2 d_H X}{d_S X}$$

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Geometry

(Hausdorff dimension)

Einstein Relation

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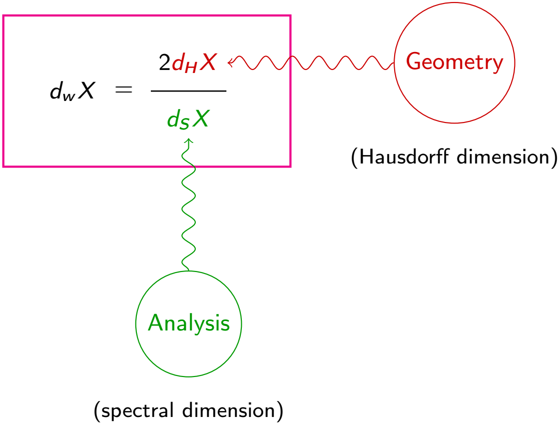

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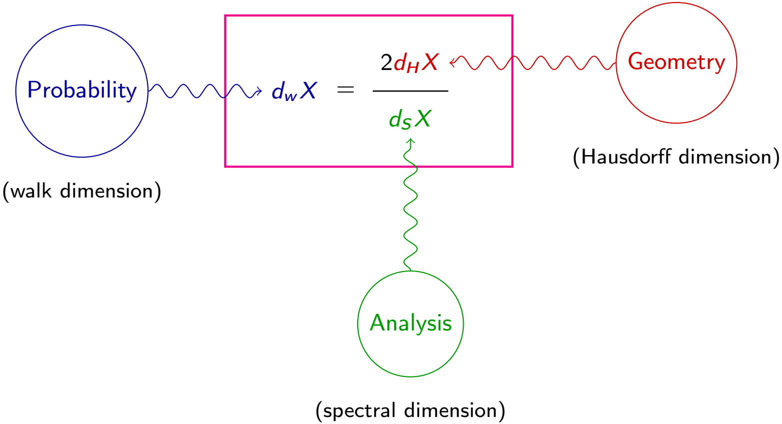
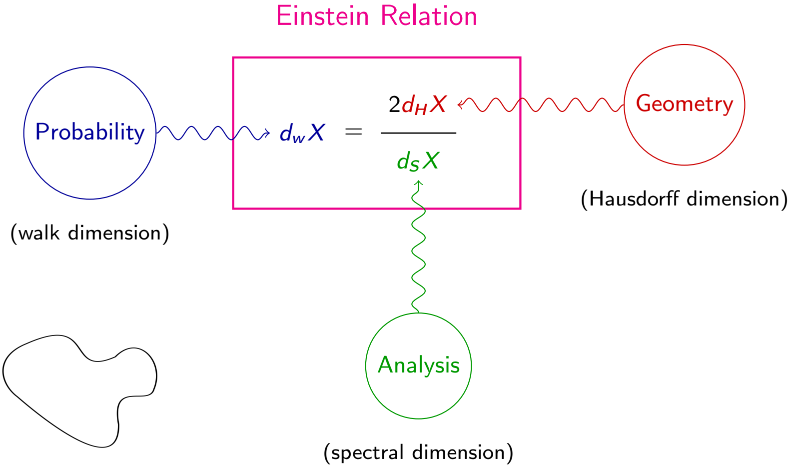
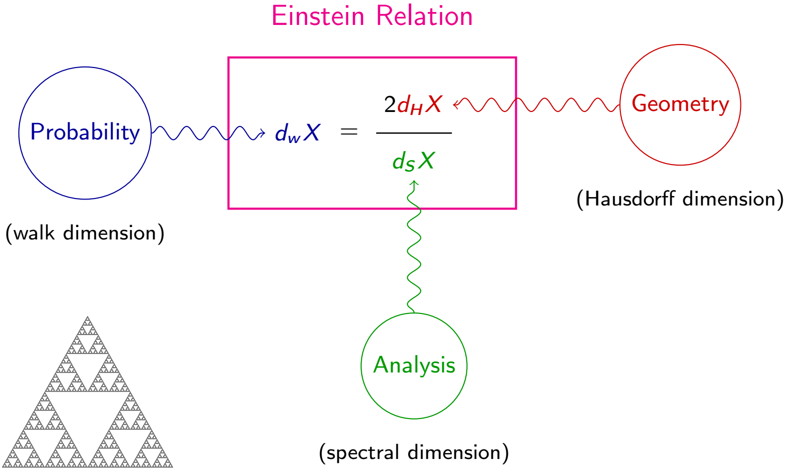



Einstein Relation
Probability
$d_w X = \frac{2 d_H X}{d_S X}$
Geometry
(walk dimension)
(Hausdorff dimension)
Analysis
(spectral dimension)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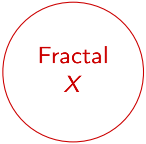
... on fractals



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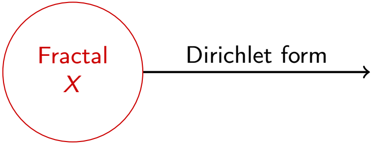

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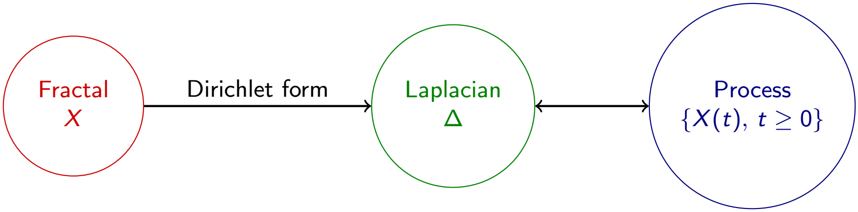

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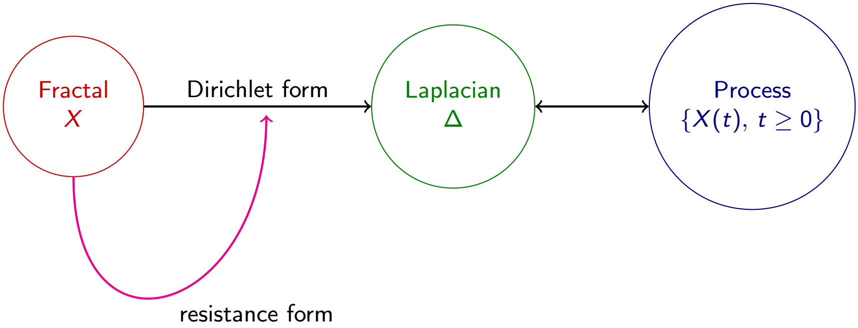
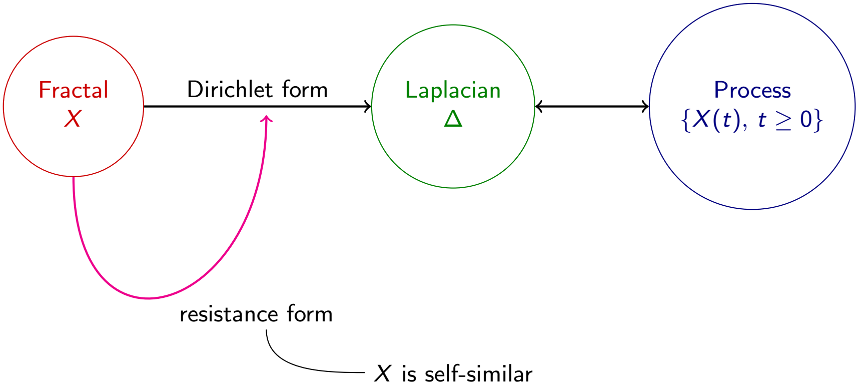

Fractal
X
Dirichlet form
Laplacian
Δ
Process
$\{X(t),\ t \geq 0\}$
resistance form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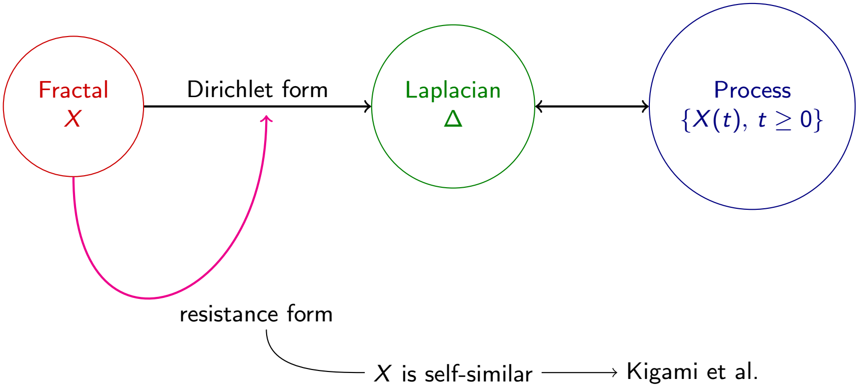
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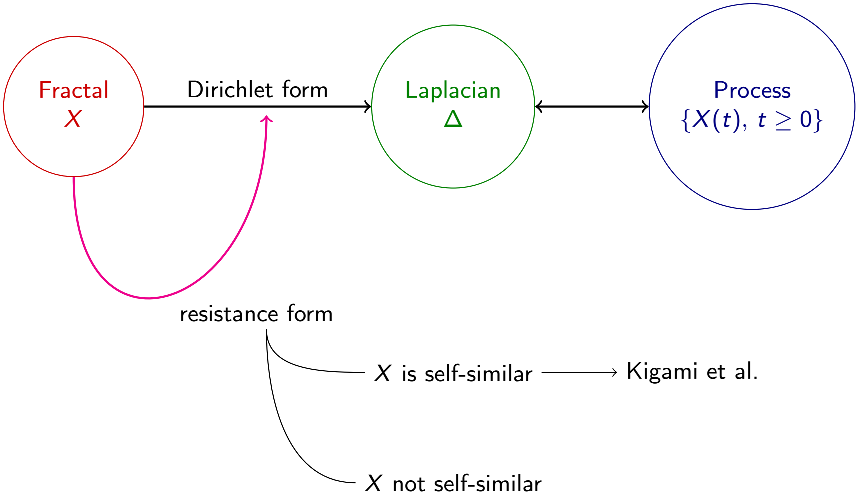
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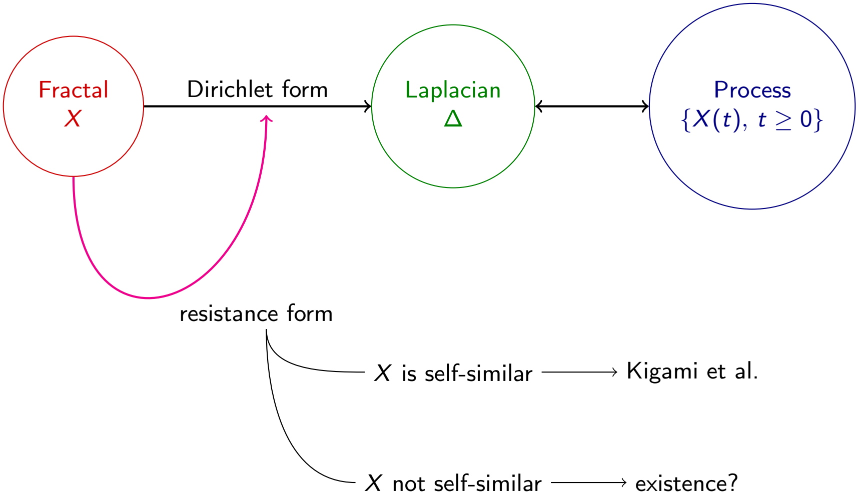
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existence?

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- A *graph* (V, E, ∂) is a finite or countable set of vertices V along with a set of edges E joining them and a map $\partial \colon E \to V \times V$ given an orientation to the edges.

Quantum graphs

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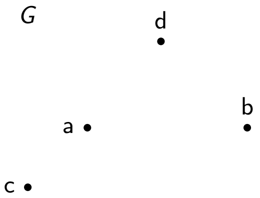

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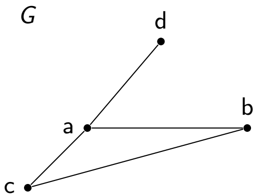

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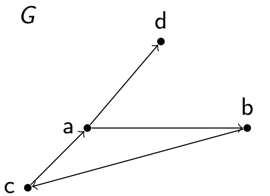

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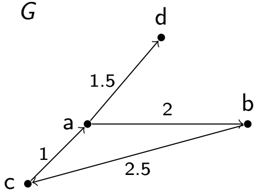

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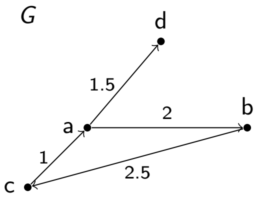

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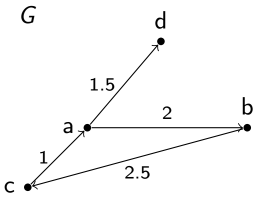

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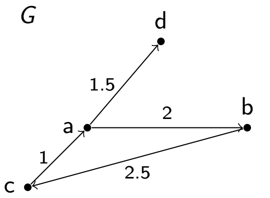


$$L^2(G) := \bigoplus_{e \in E} L^2([0, r(e)], dx)$$

$$H^1(G) := \bigoplus_{e \in E} H^1([0, r(e)], dx)$$

Definition 1.2.

Let (X, d) be a locally compact metric space and μ be a locally finite and regular measure on X. Moreover, let $\mathcal{E}\colon \mathcal{D} \to \mathcal{D}$ is a non-negative symmetric bilinear form. $(\mathcal{E}, \mathcal{D})$ is a *Dirichlet form* iff

(i) $\mathcal{D}$ is a dense subspace of $L^2(X, \mu)$,

(ii) $(\mathcal{D}, \mathcal{E}_1^{1/2})$ is a Hilbert space, where $\mathcal{E}_1(f, f) := \mathcal{E}(f, f) + (f, f)_2$,

(iii) for all $f \in \mathcal{D}$, $\overline{f} \in \mathcal{D}$ and $\mathcal{E}(\overline{f}, \overline{f}) \leq \mathcal{E}(f, f)$, where $\overline{f} := 0 \wedge f \vee 1$.

Theorem 1.3.

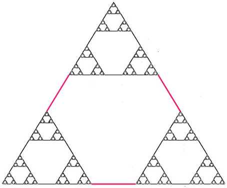
Let $(\mathcal{E}, \mathcal{D})$ be a Dirichlet form on $L^2(X, \mu)$. Then, there exists a unique non-negative self-adjoint operator $\Delta_\mu\colon \operatorname{Dom}\Delta_\mu \to L^2(X, \mu)$ such that $\operatorname{Dom}\Delta_\mu$ is dense in $L^2(X, \mu)$ and

$$\mathcal{E}(f, g) = -\int_X \Delta_\mu f g \, d\mu \qquad \forall\, g \in \mathcal{D}.$$

Δ_μ is called the Laplacian associated to $(\mathcal{E}, \mathcal{D})$.

Hanoi-type metric graphs

X Hanoi attractor of parameter α, $(0 < \alpha < \frac{1}{3})$



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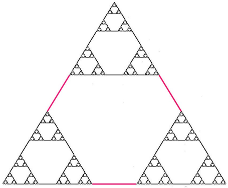

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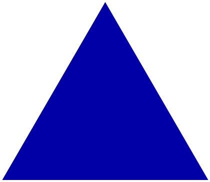

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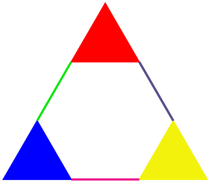

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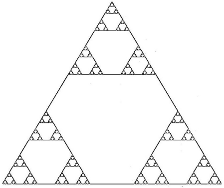

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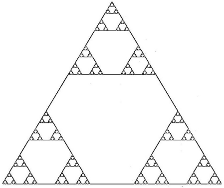

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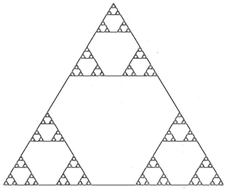

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For each $n \in \mathbb{N}_0$, the metric graph $X_n := (V_n, E_n, \partial)$ has the vertex set

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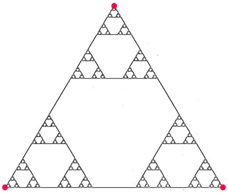

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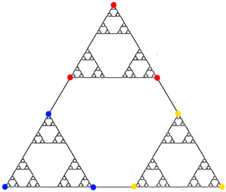

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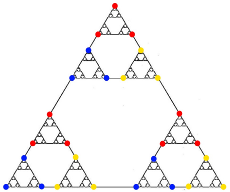

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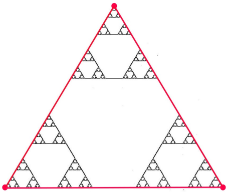

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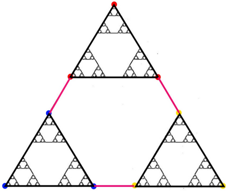

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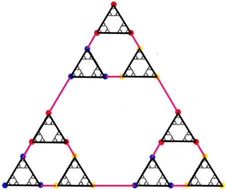


Definition 1.4.

For each $n \in \mathbb{N}_0$, the metric graph $X_n := (V_n, E_n, \partial)$ has the vertex set

$$V_n := \bigcup_{w \in \mathcal{A}^n} F_{w_1} \circ \cdots F_{w_n}(\{p_1, p_2, p_3\})$$

and the edge set $E_n := T_n \cup J_n$ given by

$$T_n := \{\{x, y\} \,|\, \exists\, w \in \mathcal{A}^n \text{ s.t. } x, y \in F_w(V_0)\}$$
$$J_n := \{\{x, y\} \,|\, \exists 0 < k < n, w \in \mathcal{A}^{k-1} \text{ s.t. } x = F_{wj}(p_i), y = F_{wi}(p_j),\ i, j \in \mathcal{A}, i \neq j\},$$

equipped with the Euclidean metric.

Definition 1.

Let $n \in \mathbb{N}_0$ and define

$$\mathcal{F}_n := \left\{ f\colon X \to \mathbb{R} \mid \forall\, n \in \mathbb{N}_0,\ f_{|X_n} \in H^1(X_n) \text{ and } f_{|e} \equiv c_e\ \forall\, e \in X \setminus \bigcup_{e \in J_n} \Phi_e([0, r(e)]) \right\},$$

where c_e is some constant that only depends on e. The non-negative symmetric bilinear form given by

$$\mathcal{E}(f,g) := \int_X f'g'\,dx \qquad f,g \in \mathcal{F}_n$$

is called the *energy of the n-th approximation of* X.

Proposition 2.1.

Let $n \in \mathbb{N}_0$ *and define the non-negative symmetric bilinear form* $\mathcal{E}_n\colon F_n \times F_n \to \mathbb{R}$ *by*

$$\mathcal{E}_n(f,f) := \inf\{\mathcal{E}(g,g) \mid g \in \mathcal{F}_k\ \forall\ k = 1,2,\ldots,n,\ \ g_{|X_n} \equiv f\}.$$

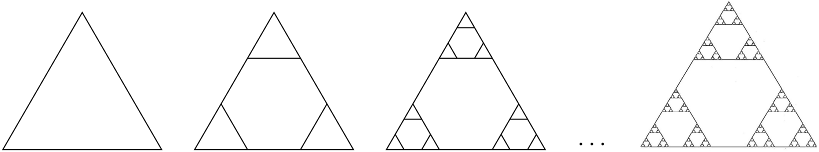
For all $n \in \mathbb{N}_0$, $(\mathcal{E}_n, F_n)$ *is a resistance form.*

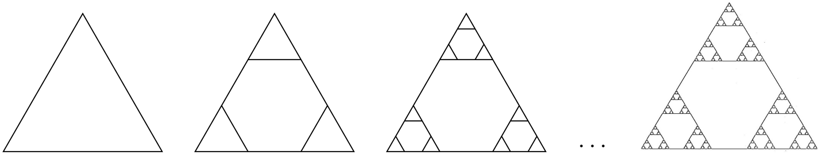
$(\mathcal{E}_n, F_n)$ resistance form $+$ Lebesgue	$\longrightarrow$	$(\mathcal{E}_n, \mathcal{D}_n)$ Dirichlet form on $L^2(X_n)$	$\longrightarrow$	Δ_n Laplacian on X_n

Definition 2.2.

For each $n \in \mathbb{N}_0$, the metric graph X_n together with the Laplacian Δ_n is a *Hanoi-type quantum graph*.

Resistance form and measure on X





Proposition 2.3.

Let us define

$$\mathcal{F} := \{f : X \to \mathbb{R} \mid \lim_{n\to\infty} \mathcal{E}_n(f_{|X_n}, f_{|X_n}) < \infty\}.$$

The pair $(\mathcal{E}, \mathcal{F})$ is a regular resistance form.

Dirichlet form and Laplacian on X

$(\mathcal{E}, \mathcal{F})$ regular resistance form

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Δ_μ^N associated Laplacian on X

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Δ_μ^N associated Laplacian on X

The same holds for the Dirichlet form with $\mathcal{D}_0 := \{f \in \mathcal{D} \mid f_{|_{\mathbf{V_0}}} \equiv 0\}$ and $\mathcal{E}_0 := \mathcal{E}_0|_{\mathcal{D}_\mathbf{0} \times \mathcal{D}_\mathbf{0}}$ and its associated Laplacian Δ_μ^D.

Eigenvalue counting function

Remark 2.4.

The spectrum of $-\Delta_\mu^{N/D}$ is a countable set where all eigenvalues are non-negative with finite multiplicity and the only accumulation point is $+\infty$.

Definition 2.5.

The eigenvalue counting function of $-\Delta_\mu^{N/D}$ at $x \geq 0$ is given by

$$N_{N/D}(x) := \#\{\lambda \in \sigma(-\Delta_\mu^{N/D}) \mid \lambda \leq x\}$$

according to multiplicity.

Definition 2.6.

The *spectral dimension* of X is the positive number $d_S X$ such that

$$N_{N/D}(x) \sim x^{\frac{d_S X}{2}}.$$

Proposition 3.1.

Let $r := \frac{1-\alpha}{2}$ *and* $s := \frac{1-\beta}{3}$*. There exist constants* $C_1, C_2 > 0$ *and* $x_0 > 0$ *such that*

(i) *if* $0 < rs < \frac{1}{9}$*, then*

$$C_1 x^{\frac{1}{2}} \le N_D(x) \le N_N(x) \le C_2 x^{\frac{1}{2}},$$

(ii) *if* $rs = \frac{1}{9}$*, then*

$$C_1 x^{\frac{1}{2}} \log x \le N_D(x) \le N_N(x) \le C_2 x^{\frac{1}{2}} \log x,$$

(iii) *if* $rs > \frac{1}{9}$*, then*

$$C_1 x^{\frac{\log 3}{-\log(rs)}} \le N_D(x) \le N_N(x) \le C_2 x^{\frac{\log 3}{-\log(rs)}}$$

for all $x > x_0$*.*

Theorem 3.2.

The spectral dimension of X is given by

$$d_S X = \begin{cases} 1, & 0 < rs \leq \frac{1}{9}, \\ \frac{\log 9}{-\log(rs)}, & \frac{1}{9} < rs < 1. \end{cases}$$

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Corollary

Suppose that the Einstein Relation holds.

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Corollary

Suppose that the Einstein Relation holds.

- The walk dimension of X is given by

$$d_w X = \begin{cases} 2, & 0 < rs \leq \frac{1}{9}, \\ 1 + \frac{\log s}{\log r}, & \frac{1}{9} < rs < 1. \end{cases}$$

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Superconductor??

Thank you for your attention!

measure on X

Let $\frac{1}{9} < \beta < \frac{1}{3}$ and set $s := \frac{1-3\beta}{3}$.

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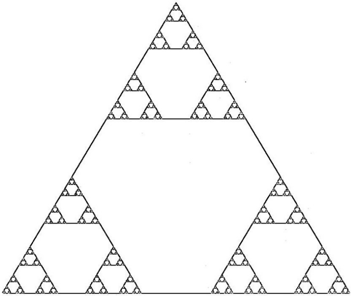

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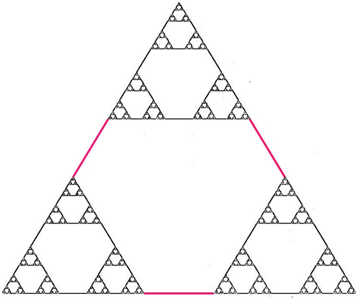
- For each $e \in J_1$, $\mu(I_e) = \beta$



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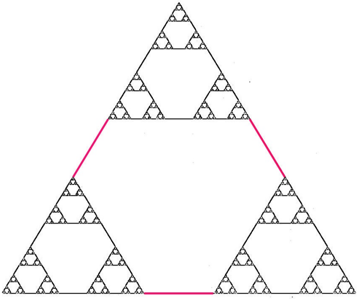
- For each $w_1 w_2 \dots w_n \in \mathcal{A}^n$, $n \in \mathbb{N}_0$
  $\mu(F_{w_1} \circ F_{w_2} \circ \cdots F_{w_n}(X)) = s^{|w|}$



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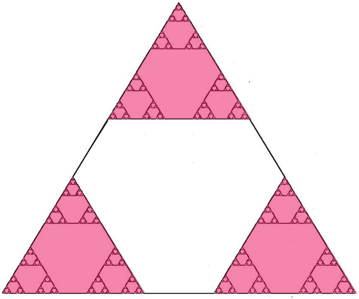

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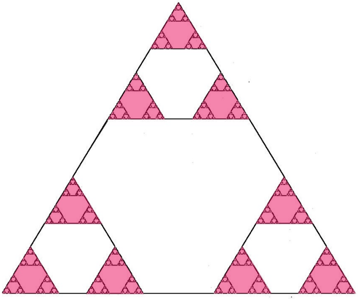

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