The Dirichlet Problem and Feller Semigroups

Wolfgang Arendt

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Maximum Principle $\Rightarrow$ there exists at most one solution.

Definition

Ω is called *Dirichlet regular* if for all $g \in C(\partial\Omega)$ there is a solution u of (P_g)

Weakly harmonic functions

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$$\langle \Delta f, \varphi \rangle := \int_\Omega f \Delta \varphi \qquad (\varphi \in \mathcal{D}(\Omega))$$

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Theorem (Weyl)

$\Delta f = 0 \quad \Rightarrow \quad f \in C^\infty(\Omega).$

The Sobolev Space $H^1(\Omega)$

$$H^1(\Omega) := \{u \in L^2(\Omega),\ \exists D_j u \in L^2(\Omega) : \\ -\int_\Omega u \frac{\partial v}{\partial x_j} = \int_\Omega (D_j u) v \quad \forall v \in \mathcal{D}(\Omega), j = 1, \dots, d\}$$

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Consistency

$$u \in C^1(\Omega) \cap L^2(\Omega),\ \tfrac{\partial u}{\partial x_j} \in L^2(\Omega) \quad \Rightarrow \quad u \in H^1(\Omega)\ \&\ D_j u = \tfrac{\partial u}{\partial x_j}$$

The Sobolev Space $H^1(\Omega)$

$H^1(\Omega)$ is a Hilbert space for the scalar product

$$(u \,|\, v)_{H^1} := \int_\Omega uv + \int_\Omega \nabla u . \nabla v,$$

where

$$\nabla u := (D_1 u, \dots, D_d u)^\top$$

$$\nabla u . \nabla v := \sum_{j=1}^{d} D_j u D_j v.$$

Poincaré Inequality

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Consequence

$$(u \,|\, v)_0 := \int_\Omega \nabla u . \nabla v$$

is an equivalent scalar product on $H_0^1(\Omega)$.

Riesz-Fréchet

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Moreover

$$\frac{1}{2}\|v\|_V^2 - Lv = \min_{w \in V}\left(\frac{1}{2}\|w\|_V^2 - Lw\right)$$

and v is the unique minimum.

The Dirichlet Problem via Riesz-Fréchet

Let $g \in C(\partial\Omega)$.

Definition

$G \in C(\overline{\Omega})$ is an H^{-1}-*extension* of g, if there exists a constant $c \geq 0$ such that

$$|\int_\Omega G\Delta\varphi| \leq c\|\varphi\|_{H^1} \qquad (\varphi \in \mathcal{D}(\Omega)),$$

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Consequence

There exists a unique $v \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla v.\nabla\varphi = \int_\Omega G\Delta\varphi \qquad (\varphi \in \mathcal{D}(\Omega))$$

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Thus $G+v$ is harmonic.
Since $v\in H_0^1(\Omega)$ and $G|_{\partial\Omega}=g$, we may consider $G+v$ as a weak solution.

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2. $u_g \in C^\infty(\Omega)$, $\Delta u_g = 0$
3. $\min_{\partial\Omega} g \leq u_g(x) \leq \max_{\partial\Omega} g \quad (x \in \Omega)$
4. *If (P_g) has a solution u, then $u = u_g$.*

Weak solutions

Open Problem

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In any case

The space F of all functions, which have an H^{-1}-extension, is dense in $C(\partial\Omega)$ and

$$T\colon F \to C(\Omega) \cap L^\infty(\Omega),\ g \mapsto u_g$$

is linear and contractive.

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Definition

u_g is called the *weak solution* of (P_g).

Perron solution

Let $g \in C(\partial\Omega)$.
$w \in C(\overline{\Omega})$ is called a *subsolution* of (P_g) if

$$-\int_\Omega w \Delta\varphi \le 0 \text{ for all } 0 \le \varphi \in \mathcal{D}(\Omega)$$

and $\limsup_{x \to z} w(x) \le g(z)$ for all $z \in \partial\Omega$.

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Theorem

$u_g(x) = \sup\{w(x),\ w \text{ is subsolution}\}$ for all $x \in \Omega$.

Variational Solution

Let $g \in C(\partial\Omega)$ with H^{-1}-extension G.
Then there is exactly one $v_g \in H_0^1(\Omega)$ such that $-\Delta v_g = \Delta G$.
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Furthermore for $v_g \in H_0^1(\Omega)$ the expression

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Let $v = u - G$.
Thus $u_g \in H_0^1(\Omega)$ is the unique minimizer of

$$\begin{cases} \frac{1}{2}\int_\Omega |\nabla (u-G)|^2 - \langle \Delta G, u - G \rangle & (u \in H^1(\Omega)) \\ & u - G \in H_0^1(\Omega). \end{cases}$$

Variational Solution

Now assume that $G \in H^1(\Omega)$. Then some terms cancel:

$$\frac{1}{2}\int_\Omega |\nabla(u-G)|^2 - \langle \Delta G, u-G\rangle = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega \nabla u.\nabla G + \frac{1}{2}\int_\Omega |\nabla G|^2 - \int_\Omega |\nabla G|^2 + \int_\Omega \nabla G.\nabla u$$

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Thus we obtain the Dirichlet principle:

Theorem

Assume that $g \in C(\partial\Omega)$ has an extension $G \in H^1(\Omega) \cap C(\overline{\Omega})$. Then

$$\min\{\int_\Omega |\nabla u|^2, \quad u \in H^1(\Omega),\ u - G \in H^1_0(\Omega)\}$$

exists and is attained exactly for u_g.

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1. $\int_\Omega |\nabla u_g|^2 < \infty$
2. *g has an extension $G \in C(\overline{\Omega}) \cap H^1(\Omega)$*

Hadamard's example

$\Omega := \{x \in \mathbb{R}^2,\ |x| < 1\}$
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Then $g \in \mathrm{C}(\partial\Omega)$ and

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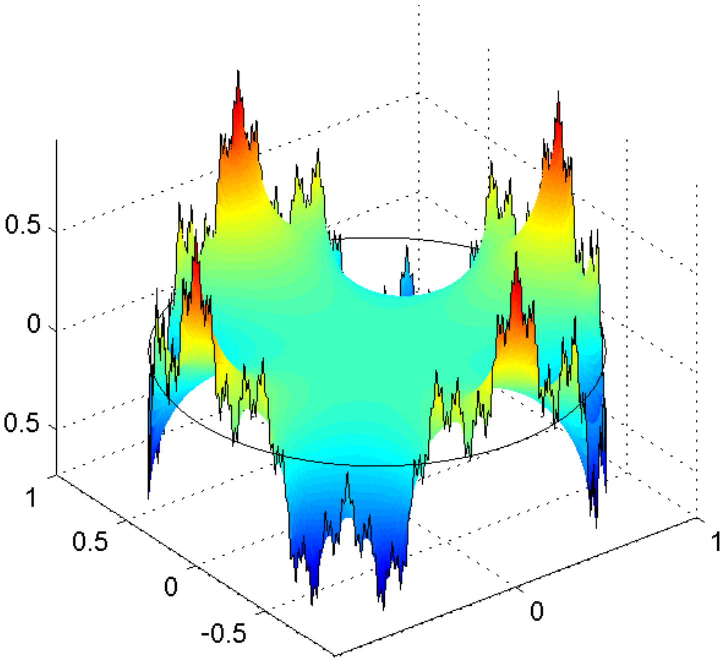
Even worse:
g does not have any extension in $\mathrm{C}(\overline{\Omega}) \cap H^1(\Omega)$.


0.5
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0
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1
0

Regular Points

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$$\begin{aligned} \Omega \text{ is } \textit{(Dirichlet-)regular} :&\Leftrightarrow \forall g \in C(\partial\Omega) : \ (P_g) \text{ has a solution} \\ &\Leftrightarrow \text{each } z \in \partial\Omega \text{ is regular} \end{aligned}$$

The weak solution via Brownian motion

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Denote by P^x the Wiener measure at $x \in \mathbb{R}^d$.

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Denote by P^x the Wiener measure at $x \in \mathbb{R}^d$.
Let $\tau_\Omega := \inf\{t > 0,\ B_t \notin \Omega\}$ be the *first hitting time.*
Let $g \in C(\partial\Omega)$. Then

$$u_g(x) = \mathbb{E}^x[g(B_{\tau_\Omega})] \qquad (x \in \Omega).$$

Probabilistic characterization of regular points

$z \in \partial\Omega$ is regular if and only if

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Thus $z \in \Omega$ is regular if and only if the Brownian motion with starting point $z \in \partial\Omega$ immediately leaves Ω.
$z \in \partial\Omega$ is regular if for all $r > 0$

$$B(r, z) \setminus \Omega \text{ is "large enough".}$$

Wiener criterion

Let $A \subset \mathbb{R}^d$.

$$\mathrm{Cap}(A) := \inf\{\|u\|^2_{H^1(\mathbb{R}^d)},\ u \in H^1(\mathbb{R}^d),$$
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Theorem (Wiener)

A point $z \in \partial\Omega$ is regular if and only if

$$\sum_{n=1}^{\infty} 2^n \,\mathrm{Cap}(B(z, 2^{-n}) \setminus \Omega) = \infty.$$

Criteria for regularity

- Ω Lipschitz $\implies$ Ω is regular.

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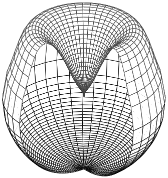

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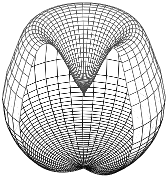
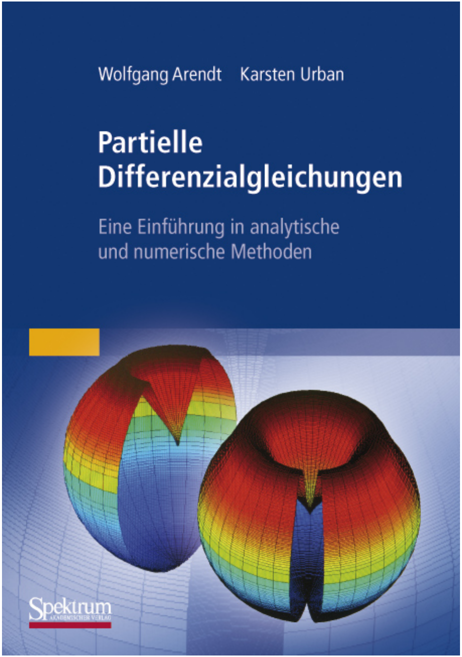


Surprising

$\exists g \in \mathrm{C}(\partial\Omega)$, g has a real analytic extension at each $z \in \partial\Omega$, but (P_g) has no solution. (W.A., D. Daners: Discrete and Continuous Dynamical Systems, 2008)


Wolfgang Arendt
Karsten Urban
Partielle Differenzialgleichungen
Eine Einführung in analytische und numerische Methoden
Spektrum

Keldy̌s

M.V. Keldy̌s *On the solvability and stability of the Dirichlet problem.*
Uspekhi Mat. Nauk 8 (1941) 144–171
Amer. Math. Soc. Translations 51 (1966) 1–73.

Regularity is non-local

Consider Lebesgue's cusp Ω with the crater at the north pole. Call the cusp point z_0 (the bottom of the crater). Then z_0 is the only singular point on the boundary.

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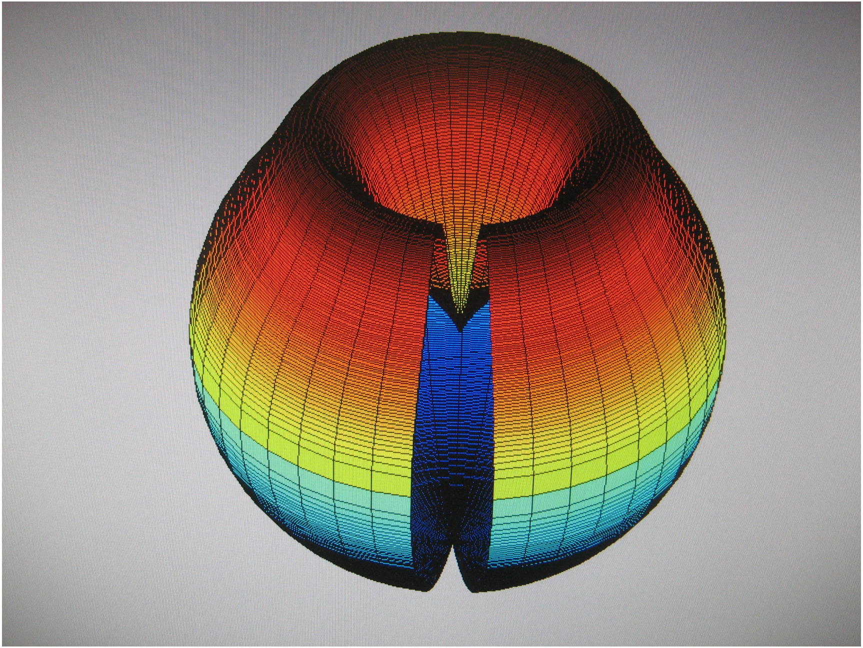
Theorem

Let U be an arbitrarily small neighborhood of the south pole. Then there exists $g \in C(\partial\Omega)$ such that $g(z) = 1$ for all $z \in \partial\Omega \setminus U$, but

$$\lim_{x \to z_0} u_g(x)$$

does not exist.



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A *semigroup* is a mapping $T\colon (0,\infty) \to \mathscr{L}(X)$ such that

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Definition

The *generator* A of T is defined by

$$D(A) := \{x \in X : \lim_{t\searrow 0} \frac{T(t)x - x}{t} \text{ exists}\}$$

$$Ax := \lim_{t\searrow 0} \frac{T(t)x - x}{t}.$$

Cauchy Problem

Consequence

For $u_0 \in D(A)$

$$u := T(\cdot)u_0 \in \mathrm{C}^1(\mathbb{R}_+, X)$$

is the unique solution of

$$\begin{cases} \dot{u}(t) = Au(t) \\ u(0) = u_0. \end{cases}$$

Feller semigroups

Let $\Omega \subset \mathbb{R}^d$ be open and bounded.
We consider the space

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with norm $\|u\|_\infty := \sup_{x\in\Omega} |u(x)|$.

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A semigroup $(T(t))_{t\geq 0}$ on $C_0(\Omega)$ is called a *Feller semigroup* if

1. $f \geq 0 \quad \Rightarrow \quad T(t)f \geq 0$, i.e. $T(t) \geq 0$.
2. $\|T(t)\| \leq 1 \qquad (t \geq 0)$

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Theorem (Phillips)

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2. $f \in D(A),\ f(x_0) = \sup_{x\in\Omega} f(x) > 0 \quad \Rightarrow \quad Af(x_0) \leq 0$
3. *There is $\lambda > 0$ such that $\lambda - A$ is surjective.*

The Dirichlet Laplacian

Recall: Let $u \in C_0(\Omega)$.

$$\Delta u \in C_0(\Omega) \quad :\Leftrightarrow \quad \exists f \in C_0(\Omega) : \int_\Omega u \Delta \varphi = \int_\Omega f \varphi \qquad (\varphi \in \mathcal{D}(\Omega))$$

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Then $w \in C(\mathbb{R}^d)$ and $\Delta w = f$.
Let $g = w|_{\partial\Omega}$. Then $u_g \in C(\overline{\Omega})$, $\Delta u_g = 0$.
Let $u = w - u_g$. Then $u \in C_0(\Omega)$, $\Delta u = f$. □

Elliptic operators in divergence form

Let $a_{ij}, b_j, c_j, c \in L^\infty(\Omega)$ be real coefficients, such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2 \qquad (\xi \in \mathbb{R}^d, x \in \Omega).$$

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$$\mathcal{A}\colon H^1_{\mathrm{loc}} \to \mathcal{D}(\Omega)'$$

$$\mathcal{A}u := \sum_{i,j=1}^{d} D_i(a_{ij}D_j u) + \sum_{j=1}^{d}(b_j D_j u + D_j(c_j u)) + cu$$

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Realization A_0 in $\mathrm{C}_0(\Omega)$:

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Assume that

$$\sum_{j=1}^{d} D_j c_j + c \leq 0,$$

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Let $b_j, c \in L^\infty(\Omega)$, $c \leq 0$, $a_{ij} \in \mathrm{C}(\overline{\Omega})$, $a_{ij} = a_{ji}$ be real coefficients, such that

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Realization of $\mathcal{A}$ in $\mathrm{C}_0(\Omega)$:

$$\begin{aligned} D(A_0) &:= \{u \in \mathrm{C}_0(\Omega) \cap W^{2,d}_{\mathrm{loc}}(\Omega) : \mathcal{A}u \in \mathrm{C}_0(\Omega)\} \\ A_0 u &:= \mathcal{A}u \end{aligned}$$

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We prove that the semigroup is even holomorphic.

Markov Processes

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Let $(T(t))_{t\geq 0}$ be a Feller semigroup on $C_0(\Omega)$, Ω a locally compact space with countable topological base.
Then there exists a strong Markov process $\{X_t,\ t\geq 0\}$ such that for all $f\in C_0(\Omega)$,

$$\begin{aligned}(T(t)f)(X_s) &= \mathbb{E}(f(X_{t+s})\,|\,X_s)\\ &= \mathbb{E}(f(X_{t+s})\,|\,\mathcal{F}_s) \qquad \text{a.e.,}\end{aligned}$$

where

$$\mathcal{F}_s = \sigma(\{X_t,\ t\leq s\}).$$

Thank you for your attention