Limit Theorems for Catalytic Branching Processes

Ekaterina Vl. Bulinskaya

Lomonosov Moscow State University

Ulm, September 2, 2013

Introduction

- L. Doering, M. Roberts (2013)
  catalytic branching process with a single catalyst
  many-to-few lemmas, renewal theory
- E.B. Yarovaya (2012)
  branching random walk with finitely many sources of particle generation
  spectral theory of operators
- E.Vl. Bulinskaya (2013)
  catalytic branching process with finitely many catalysts (CBP)
  auxiliary multi-type Bellman-Harris process

Introduction

- L. Doering, M. Roberts (2013)
  catalytic branching process with a single catalyst
  many-to-few lemmas, renewal theory
- E.B. Yarovaya (2012)
  branching random walk with finitely many sources of particle generation
  spectral theory of operators
- E.Vl. Bulinskaya (2013)
  catalytic branching process with finitely many catalysts (CBP)
  auxiliary multi-type Bellman-Harris process

Introduction

- L. Doering, M. Roberts (2013)
  catalytic branching process with a single catalyst
  many-to-few lemmas, renewal theory
- E.B. Yarovaya (2012)
  branching random walk with finitely many sources of particle generation
  spectral theory of operators
- E.Vl. Bulinskaya (2013)
  catalytic branching process with finitely many catalysts (CBP)
  auxiliary multi-type Bellman-Harris process

Description of CBP

- particles move according to an irreducible Markov chain $\eta = \{\eta(t), t \geq 0\}$ having the state space S and generated by Q-matrix A
- they branch at the presence of catalysts, $W = \{w_1, \ldots, w_N\} \subset S$ is the catalysts set
- having hit w_k ($k = 1, \ldots, N$), a particle, after an exponentially distributed time with parameter 1, either produces a random number of offsprings ξ_k with probability α_k or leaves w_k with probability $1 - \alpha_k$
- newly born particles behave as independent copies of their parent

Description of CBP

- particles move according to an irreducible Markov chain $\eta = \{\eta(t), t \geq 0\}$ having the state space S and generated by Q-matrix A
- they branch at the presence of catalysts, $W = \{w_1, \ldots, w_N\} \subset S$ is the catalysts set
- having hit w_k ($k = 1, \ldots, N$), a particle, after an exponentially distributed time with parameter 1, either produces a random number of offsprings ξ_k with probability α_k or leaves w_k with probability $1 - \alpha_k$
- newly born particles behave as independent copies of their parent

Description of CBP

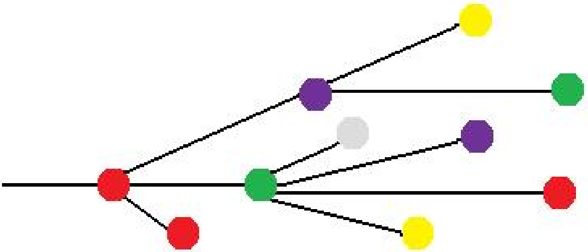
- particles move according to an irreducible Markov chain $\eta = \{\eta(t), t \geq 0\}$ having the state space S and generated by Q-matrix A
- they branch at the presence of catalysts, $W = \{w_1, \ldots, w_N\} \subset S$ is the catalysts set
- having hit w_k ($k = 1, \ldots, N$), a particle, after an exponentially distributed time with parameter 1, either produces a random number of offsprings ξ_k with probability α_k or leaves w_k with probability $1 - \alpha_k$
- newly born particles behave as independent copies of their parent

Description of CBP

- particles move according to an irreducible Markov chain $\eta = \{\eta(t), t \geq 0\}$ having the state space S and generated by Q-matrix A
- they branch at the presence of catalysts, $W = \{w_1, \ldots, w_N\} \subset S$ is the catalysts set
- having hit w_k ($k = 1, \ldots, N$), a particle, after an exponentially distributed time with parameter 1, either produces a random number of offsprings ξ_k with probability α_k or leaves w_k with probability $1 - \alpha_k$
- newly born particles behave as independent copies of their parent

Multi-type Bellman-Harris Processes

- A particle of type i has a life-length with distribution G_i.
- Just before the death the particle of type i produces a random number of offsprings according to a generating function g_i, $i = 1, \ldots, L$.



Method of CBP study

V.A. Vatutin, V.A. Topchii, E.B. Yarovaya (2003)
auxiliary Bellman-Harris process (BHP)
with two types of particles for study of a
branching random walk with a single catalyst

E.Vl.Bulinskaya (2013)
auxiliary Bellman-Harris process with
$\leq N(N+1)$ types of particles for analysis
of a catalytic branching process with N catalysts

Auxiliary Bellman-Harris Process

The particles located at time t at w_i in CBP are the particles of type i in BHP.

Each particle in CBP that has left w_j at least once within time interval $[0, t]$ and upon the last leaving w_j has not yet reached W by time t but eventually will hit w_k before possible hitting $W \setminus \{w_k\}$ is of the $(jN + k)$-th type in BHP.

We have constructed a Bellman-Harris process with $\leq N(N + 1)$ types of particles.

Classification of CBP

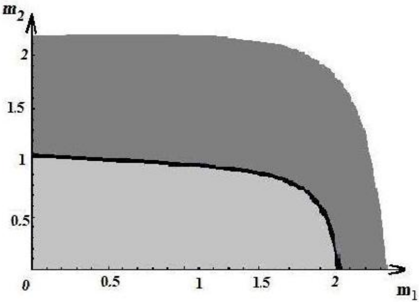
Let $M = (m_{ij})$ be the mean matrix of BHP, i.e.
m_{ij} is the mean number of the offsprings of type j produced by a particle of type i.

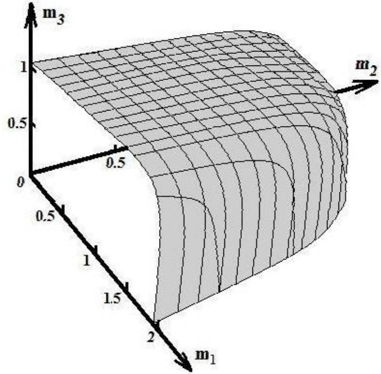
M is an irreducible matrix. Therefore, according to the Perron-Frobenius theorem
M has a positive eigenvalue $\rho(M)$ with maximal modulus which is called the Perron root.

Definition by E.Vl. Bulinskaya (2013)
CBP is called supercritical, critical or subcritical if $\rho(M) > 1$, $\rho(M) = 1$ or $\rho(M) < 1$, respectively.

Structure of the Criticality Set

For $\mathbb{E}\xi_i = m_i$, $i = 1, \ldots, N$,
the criticality set is
$C = \{(m_1, \ldots, m_N) \in \mathbb{R}_+^N : \rho(M) = 1\}$.





Notation

Let $\mu(t;y)$ be the number of particles at site y at time t in CBP. In other words, $\mu(t;y)$ is the local particles number.

Set $\mu(t) = \sum_{y\in S}\mu(t;y)$, i.e. $\mu(t)$ is the total number of particles at time t.

Put also

$$m_n(t;x,y)=\mathbb{E}_x\mu(t;y)(\mu(t;y)-1)\ldots(\mu(t;y)-n+1),$$

$$M_n(t;x)=\mathbb{E}_x\mu(t)(\mu(t)-1)\ldots(\mu(t)-n+1).$$

Main results

Theorem (E.Vl. Bulinskaya (2013))

Let $x, y \in S$, $n \in \mathbb{N}$ and $t \to \infty$. If $\rho(M) > 1$ then for some $\lambda > 0$ and $a_n(x,y)$, $A_n(x) > 0$ one has

$$m_n(t;x,y) \sim a_n(x,y)e^{\lambda t}, \quad M_n(t;x) \sim A_n(x)e^{\lambda t}.$$

If $\rho(M) = 1$ then

$$m_n(t;x,y) \sim b_n(x,y)t^{n-1}, \; M_n(t;x) \sim B_n(x)t^{2n-1}.$$

If $\rho(M) < 1$ then

$$m_n(t;x,y) \to 0, \quad M_n(t;x) \to C_n(x),$$

for some $b_n(x,y)$, $B_n(x)$, $C_n(x) \geq 0$.

Theorem (E.Vl. Bulinskaya (2013))

Let matrix A have uniformly bounded elements, $\rho(M) > 1$ *and* $\mathbb{E}\xi_i^2 < \infty$ *for each* $i = 1, \dots, N$.
Then the following relations hold true

$$\mu(t;y)e^{-\lambda t} \to \nu(y), \quad \mu(t)e^{-\lambda t} \to \nu \quad a.s.,$$

as $t \to \infty$, *where* $\nu(y)$, $y \in S$, *and* ν *are certain non-trivial nonnegative random variables.*

To prove the theorems we employ
results for the Bellman-Harris processes
(K.S. Crump (1970), Ch.J. Mode (1971),
N. Kaplan (1975))
and hitting times under taboo.

Hitting Times under Taboo

Let τ_x be the first exit time from x on the set $\{\eta(0) = x\}$ and let H be the taboo set, $H \subset S$. The hitting time of y under the taboo H is ${}_H\tau_{x,y}$ defined on the set $\{\eta(0) = x\}$ as $\inf\{t \geq \tau_x : \eta(t) = y,\ \eta(u) \notin H,\ \tau_x < u < t\}$ (as usual, $\inf\{t \in \varnothing\} = \infty$).

K.L. Chung (1960),
R.L. Tweedie (1974),
J. Kemeny, L. Snell, A. Knapp, D. Griffeath (1976),
A.M. Zubkov (1980),
R. Syski (1992), ...,
E.Vl. Bulinskaya (2013).