Around the Central Limit Theorem

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Workshop "Stochastics, Geometry and Analysis"
Ulm, September 2–6

Overview

1. Introduction
2. Dependence conditions for random field
3. The Newman conjecture. Counterexamples
4. Necessary and sufficient conditions for CLT validity for a class of random fields
5. Excursion sets
6. Convergence rates estimates in CLT

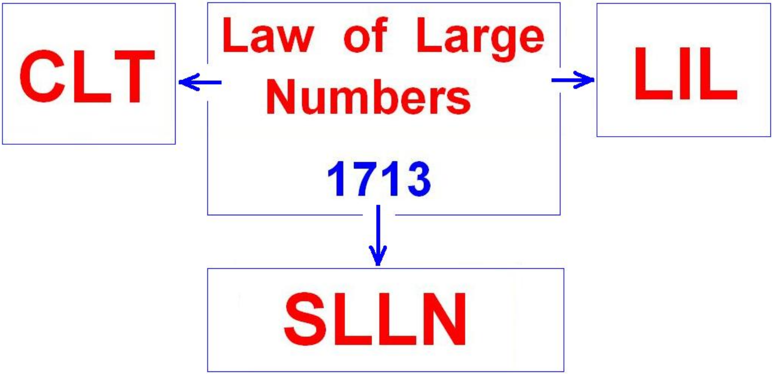
1. Introduction

Jacob Bernoulli
(1654 - 1705)


JACOBI BERNOULLI,
ARS CONJECTANDI,
TRACTATUS
DE SERIEBUS INFINITIS,
DE LUDO PILÆ
RETICULARIS.
BASILEÆ,
Impensis THURNISIORUM, Fratrum.

1713

FLUCTUATIONS OF SUMS OF INDEPENDENT RANDOM VARIABLES



Central limit theorem (CLT)

appeared in the book by Abraham de Moivre
"The Doctrine of Chances"
published in 1718, 2nd ed. 1738

Theorem (local)

Let r.v's $S_n \sim B(n,p)$, $0 < p < 1$, *i.e.*

$$\mathsf{P}(S_n = k) = \binom{n}{k} p^k q^{n-k}, \quad k = 0, \ldots, n.$$

Then for $k = np + o(n^{\frac{2}{3}})$, *as* $n \to \infty$,

$$\mathsf{P}(S_n = k) \sim \frac{1}{\sqrt{npq}} \varphi(x_k),$$

here $q = 1 - p$, $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $x_k = \frac{k-np}{\sqrt{npq}}$.

Theorem (integral)

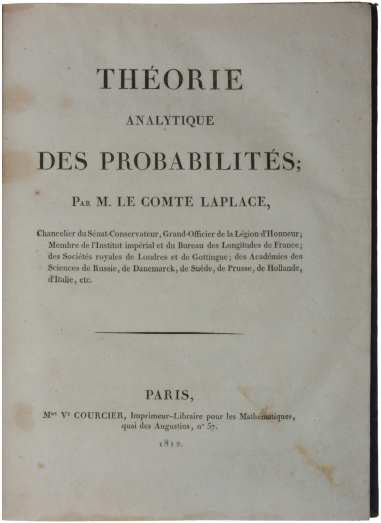
For any $-\infty < a < b < \infty$

$$\mathrm{P}\left(a \le \frac{S_n - np}{\sqrt{npq}} \le b\right) \to \int_a^b \varphi(x)dx$$

as $n \to \infty$.

$$\sum_{k: a \le \frac{k-np}{\sqrt{npq}} \le b} \mathrm{P}(S_n = k) \approx \sum_{k: a \le x_k \le b} \varphi(x_k)\Delta x_k$$

$$\Delta x_k = x_k - x_{k-1} = \frac{1}{\sqrt{npq}}.$$



THÉORIE

ANALYTIQUE

DES PROBABILITÉS;

PAR M. LE COMTE LAPLACE,

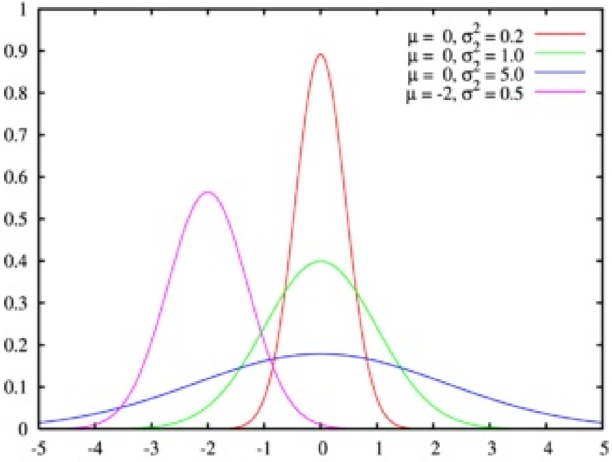
Chancelier du Sénat-Conservateur, Grand-Officier de la Légion d'Honneur; Membre de l'Institut impérial et du Bureau des Longitudes de France; des Sociétés royales de Londres et de Gottingue; des Académies des Sciences de Russie, de Danemarck, de Suède, de Prusse, de Hollande, d'Italie, etc.

PARIS,

Mme Ve COURCIER, Imprimeur-Libraire pour les Mathématiques, quai des Augustins, n° 57.

1812.

1812

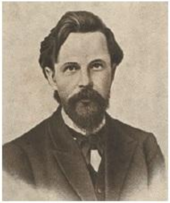


Gauss introduced the normal law in 1809

St.Petersburg school in Probability Theory

P.L.Chebyshev

(1821 – 1894)



A.A. Markov

(1856 – 1922)

A.M.Lyapunov

(1857 – 1918)

Theorem (Lyapunov)

Let, for each $n \in \mathbb{N}$, *the centered r.v's* $X_{n,1}, \dots, X_{n,k_n}$ $(k_n \in \mathbb{N})$ *be independent,* $\sum_{j=1}^{k_n} \mathsf{E}X_{\mathsf{n,j}}^2 = 1$ *and* $\mathsf{E}|\mathsf{X}_{\mathsf{n,j}}|^{\mathsf{s}} < \infty$ *for some* $s > 2$ *and any* $n \in \mathbb{N}$, $j = 1, \dots, k_n$. *If*

$$\sum_{j=1}^{k_n} \mathsf{E}|\mathsf{X}_{\mathsf{n,j}}|^{\mathsf{s}} \to 0, \quad \mathsf{n} \to \infty,$$

then

$$S_n := \sum_{j=1}^{k_n} X_{n,j} \stackrel{law}{\to} Z \sim N(0,1), \; n \to \infty.$$

Remark

For a sequence of r.v's $X_1, X_2, \ldots$ the study of normalized partial sums

$$\frac{1}{b_n}\sum_{j=1}^{n}(X_j - a_j)$$

can be reduced to analysis of arrays. Set

$$X_{n,j} = \frac{X_j - a_j}{b_n}, \quad j = 1, \ldots, n, \quad n \in \mathbb{N},$$

then $\frac{1}{b_n}\sum_{j=1}^{n}(X_j - a_j) = \sum_{j=1}^{n} X_{n,j}$.

Corollary

Let $X_1, X_2, \ldots$ *be independent r.v's such that* $\mathsf{E}|X_j|^s < \infty$ *for some* $s > 2$ *and all* $n \in \mathbb{N}$. *Set* $b_n^2 := \sum_{j=1}^n varX_j$. *If*

$$\frac{1}{b_n^s} \sum_{j=1}^n \mathsf{E}|X_j - \mathsf{E}X_j|^s \to 0, \quad n \to \infty,$$

then

$$\frac{1}{b_n} \sum_{j=1}^n (X_j - a_j) \stackrel{law}{\to} Z \sim N(0,1), \; n \to \infty.$$

Theorem (Lévy)

Let $X_1, X_2, \ldots$ be i.i.d. r.v's such that $\sigma^2 := var X_1 < \infty$. Then

$$\frac{T_n - \mathsf{E} T_n}{\sqrt{n}} \stackrel{law}{\to} Z_\sigma \sim N(0, \sigma^2), \quad n \to \infty,$$

where $T_n = \sum_{j=1}^n X_j$, $n \in \mathbb{N}$.

As usual $Z_\sigma = 0$ for $\sigma = 0$.

Theorem (Lindeberg)

Let, for each $n \in \mathbb{N}$, centered r.v's $X_{n,1}, \ldots, X_{n,k_n}$ ($k_n \in \mathbb{N}$) be independent and $\sum_{j=1}^{k_n} \mathsf{E} X_{n,j}^2 \to 1$, $n \to \infty$. If, for any $\varepsilon > 0$,

$$\sum_{j=1}^{k_n} \mathsf{E} X_{n,j}^2 \mathbb{I}\{|X_{n,j}| \geq \varepsilon\} \to 0, \quad n \to \infty,$$

then

$$S_n := \sum_{j=1}^{k_n} X_{n,j} \stackrel{law}{\to} Z \sim N(0,1), \; n \to \infty.$$

Remark

Let $X_{n,1},\dots,X_{n,k_n}$ *be an array of independent (in each row) r.v's such that* $\mathsf{E}X_{n,j}^2<\infty$ *for* $j=1,\dots,k_n,\ n\in\mathbb{N}$. *Introduce the transformation*

$$X_{n,j}\mapsto \widetilde{X}_{n,j}=(X_{n,j}-\mathsf{E}X_{\mathrm{n,j}})/\sigma_{\mathrm{n}}$$

if $\sigma_n=\left(\sum_{j=1}^{k_n} varX_{n,j}\right)^{1/2}>0$. *Then*

$$\mathsf{E}\widetilde{X}_{\mathrm{n,j}}=0,\ \ \sum_{\mathrm{j}=1}^{\mathrm{k_n}}\mathsf{E}\widetilde{X}_{\mathrm{n,j}}^2=1.$$

Corollary

Let $X_1, X_2, \ldots$ *be independent r.v's. Suppose that, for any* $\varepsilon > 0$,

$$\frac{1}{B_n^2}\sum_{j=1}^{n} \mathsf{E}(\mathsf{X_j}-\mathsf{EX_j})^2 \mathbb{I}\{|\mathsf{X_j}-\mathsf{EX_j}|\geq \varepsilon \mathsf{B_n}\} \to 0$$

as $n \to \infty$ *where* $B_n^2 = \sum_{j=1}^{k_n} \mathrm{var}\, X_{n,j} > 0$. *Then, for* $T_n := \sum_{j=1}^{n} X_j$, *one has*

$$\frac{1}{B_n}(T_n - \mathsf{E}\, T_n) \stackrel{law}{\to} Z \sim N(0,1), \quad n \to \infty.$$

Corollary

The Lindedberg condition implies *uniform negligibility of summands:*

$$\max_{1\le j\le k_n} \mathsf{E}X_{n,j}^2 \to 0, \quad n\to\infty,$$

because, for any $\varepsilon > 0$ *and* $j = 1,\dots,k_n$,

$$\mathsf{E}X_{n,j}^2 \le \varepsilon^2 + \mathsf{E}X_{n,j}^2 \mathbb{I}\{|X_{n,j}| \ge \varepsilon\}$$
$$\le \varepsilon^2 + \sum_{j=1}^{k_n} \mathsf{E}X_{n,j}^2 \mathbb{I}\{|X_{n,j}| \ge \varepsilon\}.$$

Theorem (Feller)

Let, for each $n \in \mathbb{N}$, *centered r.v's* $X_{n,1}, \dots, X_{n,k_n}$ $(k_n \in \mathbb{N})$ *be independent and* $\sum_{j=1}^{k_n} \mathrm{EX}_{\mathrm{n,j}}^2 = 1$. *Assume that* *uniform negligibility condition is satisfied and*

$$S_n := \sum_{j=1}^{k_n} X_{n,j} \overset{law}{\to} Z \sim N(0,1), \ \ n \to \infty.$$

Then the Lindeberg condition holds.

Theorem (Zolotarev, Rotar')

Let, for each $n \in \mathbb{N}$, *the centered r.v's* $X_{n,1}, \ldots, X_{n,n}$ *be independent and* $\sum_{j=1}^{n} \sigma_{n,j}^2 = 1$ *where* $\sigma_{n,j}^2 = \mathsf{E}X_{n,j}^2 > 0$. *Then*

$$S_n := \sum_{j=1}^{n} X_{n,j} \stackrel{law}{\to} Z \sim N(0,1), \ \ n \to \infty,$$

if and only if, for each $\varepsilon > 0$,

$$\sum_{j=1}^{n} \int_{\{|x|>\varepsilon\}} |x||F_{n,j}(x) - \Phi_{n,j}(x)|dx \to 0$$

as $n \to \infty$.

Here, for $x \in \mathbb{R}$,

$$F_{n,j}(x) = \mathsf{P}(X_{n,j} \le x)$$

and

$$\Phi_{n,j}(x) = \mathsf{P}(Z_{n,j} \le x)$$

where

$$Z_{n,j} \sim N(0, \sigma_{n,j}^2), \; j = 1, \ldots, n, \;\; n \in \mathbb{N}.$$

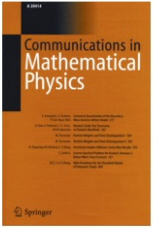
2. Dependence conditions for random field

T.E.Harris, E.L.Lehmann, J.Esary, F.Proschan, D.Walkup, C.Fortuin, P.Kasteleyn, J.Ginibre, K.Alam, K.M.L.Saxena and K.Joag-Dev introduced starting from the 60's of the last century new classes of *positively* (and later *negatively*) dependent random variables.

Mathematical Statistics, Reliability Theory, Percolation and Statistical Physics are the main sources of interest here.

The concept of association is basic.

The seminal paper by C.M.Newman (1980) gave the origin to establishing limit theorems of Probability Theory for models described by means of associated random fields or ones possessing related properties.

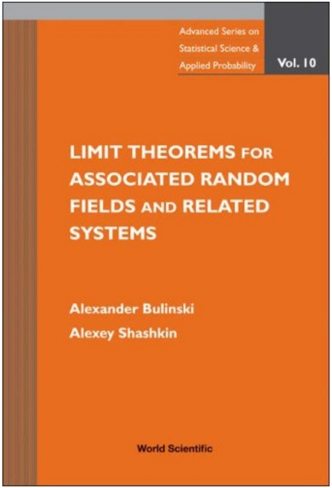


Definition (Harris, ..., Joag-Dev)

Real-valued random field $X = \{X_t, t \in T\}$ *is called* *positively associated* (*one writes* $X \in \mathrm{PA}$) *if for any finite disjoint sets* $I = \{s_1, \dots, s_m\} \subset T,\ J = \{t_1, \dots, t_n\} \subset T$ *and all bounded coordinate-wise non-decreasing Lipschitz functions* $f : \mathbb{R}^m \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$ *one has*

$$\mathrm{cov}(f(X_{s_1}, \dots, X_{s_m}), g(X_{t_1}, \dots, X_{t_n})) \geq 0. \quad (2.1)$$

A field X *is* *negatively associated* ($X \in \mathrm{NA}$) *if* " $\geq$ " в (2.1) *is replaced by* " $\leq$ ".
X *is* *associated* ($X \in \mathrm{A}$) *if* (2.1) *holds for any finite* $I, J \subset T$ *without assumption* $I \cap J = \varnothing$.


Advanced Series on
Statistical Science &
Applied Probability
Vol. 10
LIMIT THEOREMS FOR ASSOCIATED RANDOM FIELDS AND RELATED SYSTEMS
Alexander Bulinski
Alexey Shashkin
World Scientific

1. Theorem (Esary, Proschan, Walkup) Any collection of *independent random variables* is associated.

2. Theorem (Pitt) Let $X = \{X_t, t \in T\}$ be a *real-valued Gaussian random field*.

$$X \in A \Longleftrightarrow cov\,(X_s, X_t) \geq 0 \;\; s, t \in T.$$

3. Theorem (Joag-Dev, Proschan) Let $X = \{X_t, t \in T\}$ be a *real-valued Gaussian random field*.

$$X \in NA \Longleftrightarrow cov\,(X_s, X_t) \leq 0 \;\; s, t \in T, \;\; s \neq t.$$

4. <u>Theorem</u> (Bulinski, Shashkin) Let $\{x_i\}$ be a Poisson spatial process in $\mathbb{R}^d$ with intensity measure Λ. Let $\{\xi_i, i \in \mathbb{N}\}$ be i.i.d. r.v's, $\xi_1 \geq 0$ and $E\xi_1 < \infty$. Assume that $\{x_i\}$ and $\{\xi_i, i \in \mathbb{N}\}$ are independent. Let $\psi : \mathbb{R}^d \to \mathbb{R}_+$ be a Borel function such that

$$\int_{\mathbb{R}^d} \psi(t - x)\Lambda(dx) < \infty, \ \ t \in \mathbb{R}^d.$$

Then a <u>*shot-noise random field*</u>

$$X = \left\{X(t) = \sum_i \xi_i \psi(t - x_i), \ \ t \in \mathbb{R}^d\right\} \in A.$$

5. Theorem (Bulinski, Shashkin)

Let $\{M_i, i \in \mathbb{N}\}$ be i.i.d. random measures on $\mathbb{R}^d$ and $\{x_i, i \in \mathbb{N}\}$ a spacial Poisson process in $\mathbb{R}^d$. Assume that $\{M_i, i \in \mathbb{N}\}$ and $\{x_i, i \in \mathbb{N}\}$ are independent. Then the cluster point random field

$$X = \{X(B) = \sum_i M_i(B + x_i),\ B \in \mathcal{B}(\mathbb{R}^d)\} \in A.$$

Such random fields were used by J.Neyman and E.Scott in astrophysical models.

6. Theorem (Burton, Waymire, Evans)

Any infinitely divisible random measure defined on a Polish space S is associated.

Definition

A random measure M defined on $(S,\mathcal{B})$ is called infinitely divisible if, for any $k \in \mathbb{N}$, there exist (on some $(\Omega,\mathcal{F},\mathsf{P})\times(S,\mathcal{B})$) i.i.d. random measures $M_{k,1},\dots,M_{k,k}$ such that

$$Law(M) = Law(M_{k,1} + \ldots + M_{k,k}).$$

Definition

A random vector $Y = (Y_1, \ldots, Y_n)$ *with values in* $\mathbb{R}^n$ *is called* ***stable*** *if, for each* $k \in \mathbb{N}$*, there exist independent copies* $Y_k^{(1)}, \ldots, Y_k^{(k)}$ *of* Y*, non-random* $a(k) \in \mathbb{R}$ *and* $b(k) \in \mathbb{R}^n$ *such that*

$$Law\left(Y_k^{(1)} + \ldots + Y_k^{(k)}\right) = Law(a(k)Y + b(k)).$$

One can prove that $a(k) = k^{\alpha}$ where $\alpha \in (0, 2]$. Thus one says that Y is **α-stable**.

7. Theorem (Lee, Rachev, Samorodnitsky)

Let Y be an α-stable vector in $\mathbb{R}^n$ where $\alpha \in (0, 2)$. Then $\{Y_1, \dots, Y_n\} \in A$ if and only if $\Gamma(S_-) = 0$ where

$$S_- = \{(s_1, \dots, s_n) \in \mathbb{S}^{n-1} : \; s_i s_j < 0 \text{ for some } i, j.$$

Here Γ is the spectral measure of Y and $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$. $\{Y_1, \dots, Y_n\} \in NA$ if and only if $\Gamma(S_+) = 0$ where

$$S_+ = \{(s_1, \dots, s_n) \in \mathbb{S}^{n-1} : \; s_i s_j > 0 \text{ for } i \neq j.$$

Let $(L, \leq_L)$ be a partially ordered space. A function $f : L \to \mathbb{R}$ is called $\leq_L$-increasing if

$$x, y \in L, \quad x \leq_L y \Longrightarrow f(x) \leq f(y).$$

Definition (Lindqvist)

A probability measure μ defined on a space $(L, \mathcal{B}, \leq_L)$ is called associated (one writes $\mu \in A$ or $(L, \mathcal{B}, \leq_L, \mu) \in A$) if, for any bounded $\leq_L$-increasing $\mathcal{B}|\mathcal{B}(\mathbb{R})$-measurable functions $f : L \to \mathbb{R}$ and $g : L \to \mathbb{R}$,

$$\int_L fg \, d\mu \geq \int_L f \, d\mu \int_L g \, d\mu.$$

Consider a stochastic differential equation

$$\begin{cases} dX_t = b(X_t)dt + \sigma(X_t)dW_t, \\ X_0 = x_0, \end{cases} \qquad (2.2)$$

studied on a Wiener space
$(\Omega, \mathcal{F}, \mathrm{P}) = (C_0[0,T], \mathcal{B}(C_0[0,T]), \mathbb{W})$
endowed with a partial order $\le_{inc}$.

$$x \le_{inc} y \text{ if } x(t) - x(s) \le y(t) - y(s)$$

for any $0 \le s \le t \le T$.

8. Theorem (Barbato) Let $b : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function and the same be true of the derivative σ' of $\sigma : \mathbb{R} \to [\varepsilon, M]$ for some $\varepsilon, M > 0$. Then there exists a stochastic process $X = \{X_t(\omega), t \in [0, T], \omega \in \Omega\}$ such that

(i) X is a strong solution of (2.2);

(ii) for each $t \in [0, T]$, $X_t(\cdot)$ is an increasing function on $(\Omega, \mathcal{F}, \mathsf{P}, \leq_{inc})$.

(iii) $X \in A$.

Remark

Let $X = \{X_t, t \in T\}$ be a family of random variables. Then $X \in A \iff Law(X) \in A$.

9. Theorem (Fortuin, Kasteleyn, Ginibre)

Let L be finite distributive lattice and μ a probability measure on $(L, 2^L)$ such that, for any $x, y \in L$,

$$\boxed{\mu(x \vee y)\mu(x \wedge y) \geq \mu(x)\mu(y).}$$

Then $\mu \in A$.

The Ising ferromagnet is a finite set of particles having spin either 1 or -1 and the corresponding energy of a configuration being
$$E(\sigma) = \sum\nolimits_{i,j \in V} J_{i,j}\sigma_i\sigma_j + \sum\nolimits_{i \in V} \mu_i \sigma_i$$
where $\sigma = \{\sigma_i, i \in V\}$, $J_{i,j}, \mu_i \in \mathbb{R}$, $i, j \in V$.

$$\mathsf{P}(X_i = \sigma_i, i \in V) = \frac{1}{Z_T} \exp\left\{-\frac{E(\sigma)}{T}\right\},$$

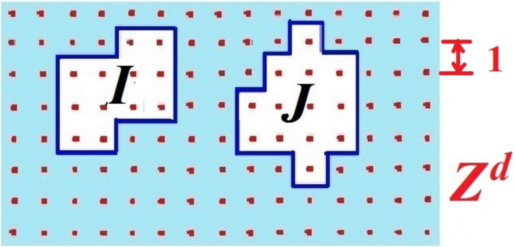
$$Z_T = \sum_{\sigma \in \{-1,1\}^{\sharp V}} \exp\left\{-\frac{E(\sigma)}{T}\right\}, \quad T > 0.$$

Further extensions

Definition (Bulinski, Suquet)

Let $X = \{X(t), t \in \mathbb{Z}^d\}$ *be a random field such that* $\mathsf{E}X(t)^2 < \infty$ *for* $t \in \mathbb{Z}^d$. *One says that* X *is* *quasi-associated* ($X \in \mathsf{QA}$) *if, for any disjoint finite sets* $I, J \subset \mathbb{Z}^d$ *and bounded Lipschitz functions* $f : \mathbb{R}^{\sharp I} \to \mathbb{R}$, $g : \mathbb{R}^{\sharp J} \to \mathbb{R}$,

$$|cov(f(X_I), g(X_J))| \le Lip(f) Lip(g) \sum_{s \in I} \sum_{t \in J} |cov(X(s), X(t))|. \quad (2.3)$$



For $f : \mathbb{R}^n \to \mathbb{R}$ set

$$Lip(f) = \sup_{x,y\in\mathbb{R}^n, x\neq y} \frac{|f(x)-f(y)|}{\|x-y\|_1},$$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \quad x = (x_1, \dots, x_n).$$

10. Theorem (Bulinski, Shabanovich) If $X = \{X_t, t \in T\}$ is a random field such that $\mathsf{E}X_t^2 < \infty$, $t \in T$, then $X \in \mathbf{QA}$ whenever $X \in PA$ or $X \in NA$.

11. Theorem (Louhichi, Shashkin) Let $X = \{X_t, t \in T\}$ be a Gaussian random field (with covariance function taking in general positive and negative values) then $X \in \mathbf{QA}$.

Definition (Bulinski, Suquet)

A random field $X = \{X(t), t \in \mathbb{Z}^d\}$ *is called* (BL, θ)*-dependent* ($X \in (BL, \theta)$) *if there exists non-increasing sequence* $(\theta_r)_{r \in \mathbb{N}}$, $\theta_r \to 0$ *as* $r \to \infty$, *such that for any finite disjoint sets* $I, J \subset \mathbb{Z}^d$ *and all bounded Lipschitz (BL) functions* $f : \mathbb{R}^{\sharp I} \to \mathbb{R}$ *and* $g : \mathbb{R}^{\sharp J} \to \mathbb{R}$ *one has*

$$|cov(f(X_I), g(X_J))| \leq Lip(f) Lip(g) (\sharp I \wedge \sharp J) \theta_r, \quad (2.4)$$

where
$r = dist(I, J) := \min\{\|s - t\|_\infty : s \in I, t \in J\}$.

P.Doukhan and S.Louhichi ($d = 1$).

Let $X = \{X_t, t \in \mathbb{Z}^d\} \in$ **QA** and the Cox - Grimmett coefficient

$$u_r = \sup_{s\in\mathbb{Z}^d} \sum_{t:\, \|s-t\|_\infty \geq r} |cov(X_s, X_t)| \to 0,$$

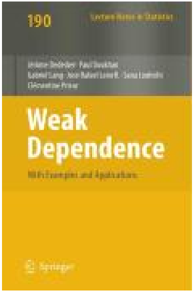
$r \to \infty$, then $X \in (BL, \theta)$ with $\theta_r = u_r$, $r \in \mathbb{N}$.

Examples

A.Shashkin showed that there are interacting particle systems possessing the (BL, θ)-dependence but which are not quasi-associated.

On models of this type we refer to the book by T.Liggett

In J.Dedecker et al. one can find other examples of random fields (for instance the autoregressive ones) having (BL, θ)-dependence property or related ones.



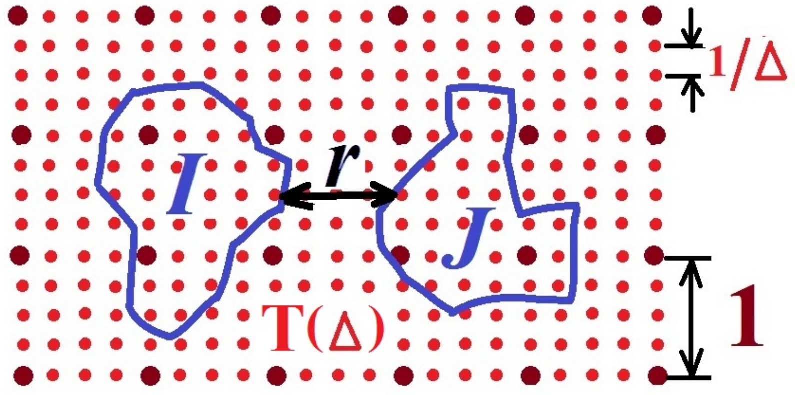
Let $T(\Delta) = \{j/\Delta : j = (j_1, \dots, j_d) \in \mathbb{Z}^d\}$, $\Delta > 0$.

Definition (Bulinski)

Real valued random field $X = \{X(t), t \in \mathbb{R}^d\} \in (BL, \theta)$ *if there exists non-increasing function* $\theta(r)$, $\theta(r) \to 0$, *as* $r \to \infty$, *such that, for all* Δ *large enough and any finite disjoint sets* $I, J \subset T(\Delta)$ *and Lipschitz functions* $f : \mathbb{R}^{\sharp I} \to \mathbb{R}$, $g : \mathbb{R}^{\sharp J} \to \mathbb{R}$ *one has*

$$|cov(f(X_I), g(X_J))| \leq Lip(f) Lip(g) (\sharp I \wedge \sharp J) \theta(r) \Delta^d \quad (2.5)$$

where $r = dist(I, J)$.


1/Δ
I
r
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T(Δ)
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Remark

If a wide-sense stationary random field $X = \{X_t, t \in \mathbb{R}^d\} \in \mathbf{QA}$ *and its covariance function* $R = R(t)$, $t \in \mathbb{R}^d$, *is directly integrable in the Riemann sense then* (2.5) *holds with*

$$\theta_X(r) = 2\int_{\|t\|_\infty \geq r} |R(t)|dt, \quad r > 0.$$

We write $\theta_X(r)$ to emphasize that $\theta(r)$ depends on X as well.

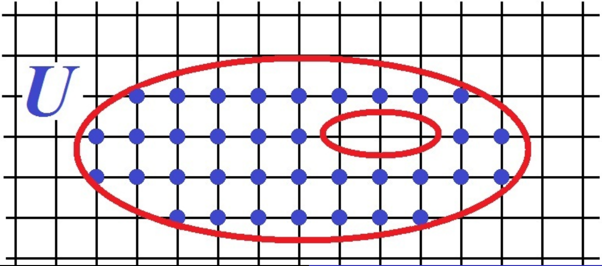
Some other research directions related to association

1. Percolation and correlation inequalities.
2. Stochastic orders and supermodular functions.
3. Markov and diffusion processes.
4. Negative dependence and networks.

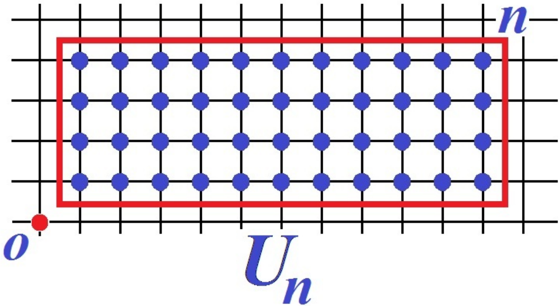
3. The Newman conjecture. Counterexamples

For random field $X = \{X_j, j \in \mathbb{Z}^d\}$ and a finite set $U \subset \mathbb{Z}^d$ introduce

$$S(U) := \sum_{j \in U} X_j.$$



If $U_n = (0, n] = (0, n_1] \times \ldots \times (0, n_d]$ where $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$



we write $S_n = S(U_n)$, $n \in \mathbb{N}^d$.

Theorem (Newman)

Let $X = \{X_j, j \in \mathbb{Z}^d\}$ *be a stationary associated random field such that*

$$\sigma^2 := \sum_{j\in\mathbb{Z}^d} cov(X_0, X_j) < \infty. \qquad (3.1)$$

Then CLT holds, namely

$$r^{-d/2}(S(C_r) - ES(C_r)) \stackrel{law}{\to} Z \sim N(0,\sigma^2), \quad r \to \infty,$$

where

$$C_r = \{j \in \mathbb{Z}^d : 1 \le j_k \le r, k = 1, \ldots, d\}, r \in \mathbb{N}.$$

If X is non-degenerate then $\sigma^2 > 0$ and one can claim that

$$\frac{S(U_n) - \mathsf{E}S(U_n)}{\sqrt{var\, S(U_n)}} \xrightarrow{law} Z \sim N(0,1), \quad n \to \infty, \qquad (3.2)$$

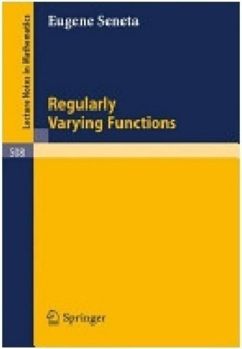
where $var\, S(U_n) \sim \sigma^2 \langle n \rangle$ (as $n \to \infty$) and $\langle n \rangle = n_1 \dots n_d$ for $n = (n_1, \dots, n_d)$.

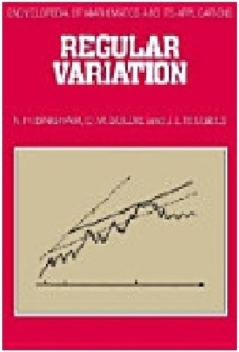
Definition (Karamata)

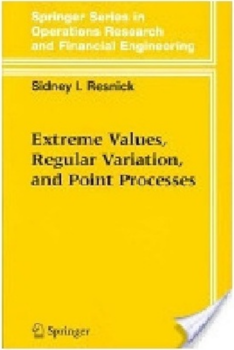
A function $L:\mathbb{R}_+^d\to\mathbb{R}$ *such that* $L(x)\neq 0$ *for all* x *large enough is called slowly varying (at infinity) if for any vector* $a=(a_1,\dots,a_d)$ *with* $a_i>0,\ i=1,\dots,d,$

$$\frac{L(a_1x_1,\dots,a_dx_d)}{L(x_1,\dots,x_d)}\to 1 \ \text{ as } \ x\to\infty, \tag{3.3}$$

i.e. $x_1\to\infty,\dots,x_d\to\infty$. *We denote the set of all such functions* $\mathcal{L}(\mathbb{R}_+^d)$. *A function* $L:\mathbb{N}^d\to\mathbb{R}$ *is called slowly varying (at infinity) if* (3.3) *holds for any* $a=(a_1,\dots,a_d)\in\mathbb{N}^d$ *and* $x\in\mathbb{N}^d$ $(x\to\infty)$. *Then we write* $L\in\mathcal{L}(\mathbb{N}^d)$.


Eugene Seneta
Regularly
Varying Functions
508
Springer


REGULAR
VARIATION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Regular Variation,
and Point Processes
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Example.

If $L(x) := \prod_{k=1}^{d} \log(x_k \vee 1)$ where $x \in \mathbb{R}_+^d$ then $L \in \mathcal{L}(\mathbb{R}_+^d)$.

Remark

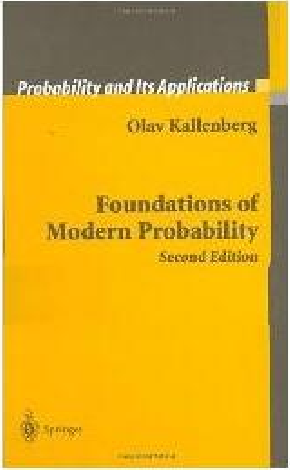
If $L \in \mathcal{L}(\mathbb{R}_+^d)$ then $L_{|\mathbb{N}^d} \in \mathcal{L}(\mathbb{N}^d)$. Not any function L belonging to $\mathcal{L}(\mathbb{N}^d)$ admits an extension to function from the class $\mathcal{L}(\mathbb{R}_+^d)$ even for $d = 1$. However if coordinate-wise nondecreasing function $L \in \mathcal{L}(\mathbb{N}^d)$ then $H(x) := L([\widetilde{x}])$ belongs to $\mathcal{L}(\mathbb{R}_+^d)$. Here $\widetilde{x} = (x_1 \vee 1, \dots, x_d \vee 1)$ for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $[x] = ([x_1], \dots, [x_d])^\top$.

Theorem (Lévy, Feller, Khinchin)

Let $X_1, X_2, \dots$ be a sequence of i.i.d. nondegenerate random variables. Then

$$a_r \sum_{k=1}^{r} (X_k - m_r) \xrightarrow{law} Z \sim N(0,1), \quad r \to \infty,$$

for some constants a_r and m_r $(r \in \mathbb{N})$ if and only if the function $L(x) := \mathsf{E}(X_1^2 \mathbb{I}\{|X_1| \le x\})$ belongs to $\mathcal{L}(\mathbb{R}_+)$. In particular such convergence holds with $a_r = r^{-1/2}$ and $m_r = 0$ $(r \in \mathbb{N})$ if and only if $\mathsf{E}X_1 = 0$ and $\mathsf{E}X_1^2 = 1$.


Probability and Its Applications
Olav Kallenberg
Foundations of
Modern Probability
Second Edition
Springer

During long time there was no answer to the **Newman conjecture** concerning the weakening of condition (3.1) appearing in CLT. Newman believed that for partial sums taken over cubes C_r instead of (3.1) it is sufficient to assume (for a stationary field $X = \{X_j, j \in \mathbb{Z}^d\} \in \mathsf{A}$, $\mathsf{E}X_0^2 < \infty$) that a function

$$\mathsf{K}(r) = \sum_{j\in\mathbb{Z}^d:\|j\|\le r} \mathrm{cov}(X_0, X_j), \quad r \in \mathbb{N}, \quad (3.4)$$

belongs to $\mathcal{L}(\mathbb{N})$ where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$. More exactly Newman conjectured that (3.2) is true for such field X and integer cubes $C_r = U_{(r,\dots,r)}$.

First countrexample to this hypothesis was constructed (for $d = 1$) by Herrndorf (1984). The general answer was given by Shashkin (2005) who demonstrated that *finite susceptibility condition* (3.1) has in a sense an optimal character.

In the paper by Bulinski and Vronski (2003) the Newman theorem was extended to partial sums over regularly growing subsets of $\mathbb{Z}^d$. Further generalizations can be found in the book by Bulinski and Shashkin (2007) (see Chapter 3).

4. Necessary and sufficient conditions for CLT

Definition

A family $X = \{X_t, t \in T\}$ of random variables is called uniformly integrable if

$$\lim_{c\to\infty} \sup_{t\in T} \mathsf{E}|X_t|\mathbb{I}\{|X_t| \geq c\} = 0.$$

Due to De la Valle-Poussen X is uniformly integrable if and only if there exists nondecreasing function $G : \mathbb{R}_+ \to \mathbb{R}_+$ such that $G(x)/x \to \infty$ as $x \to \infty$ and $\sup_{t\in T} \mathsf{E}G(|X_t|) < \infty$.

For a wide-sense stationary random field $X = \{X_j, j \in \mathbb{Z}^d\}$ introduce the function

$$\mathrm{K}_X(n) = \sum_{j\in\mathbb{Z}^d: -n\le j\le n} \mathrm{cov}\,(X_0, X_j), \quad n \in \mathbb{N}^d.$$

For $n = (n_1, \ldots, n_d)$ and $j = (j_1, \ldots, j_d)$ set
$\langle n\rangle := n_1 \ldots n_d$;
$-j \le n \le n$ means $-j_k \le n_k \le j_k$,
$k = 1, \ldots, d$.

Theorem (Bulinski)

Let $X = \{X_j, j \in \mathbb{Z}^d\} \in$ PA *be a (strictly) stationary random field such that* $0 < \mathsf{E}X_0^2 < \infty$ *and* $\mathsf{K}_X(\cdot) \in \mathcal{L}(\mathbb{N}^d)$. *Then* X *satisfies* CLT, *i.e.*

$$\frac{S_n - \mathsf{E}S_n}{\sqrt{\mathrm{var} S_n}} \xrightarrow{law} Z \sim \mathcal{N}(0,1) \text{ as } n \to \infty, \qquad (4.1)$$

if and only if the family $\{(S_n - \mathsf{E}S_n)^2/(\langle n \rangle \mathsf{K}_X(n)), n \in \mathbb{N}^d\}$ *is uniformly integrable.*

Consider now a sequence of "integer cubes" $C_r = (0, r]^d \cap \mathbb{Z}^d$, $r \in \mathbb{N}$.

Introduce the function

$$\boxed{\mathsf{K}(r) := \mathsf{K}_X(n)}$$

where $r \in \mathbb{N}$ and $n = (r, \ldots, r) \in \mathbb{N}^d$.

Theorem (Bulinski)

Let $X = \{X_j, j \in \mathbb{Z}^d\} \in$ PA *be a (strictly) stationary random field such that* $0 < \mathsf{E}X_0^2 < \infty$ *and* $\mathsf{K}(\cdot) \in \mathcal{L}(\mathbb{N})$. *Then*

$$\frac{S(C_r) - \mathsf{E}S(C_r)}{\sqrt{\mathrm{var}\, S(C_r)}} \xrightarrow{law} Z \sim \mathcal{N}(0,1), \quad r \to \infty, \quad (4.2)$$

if and only if the sequence $((S(C_r) - \mathsf{E}S(C_r))^2/(r^d\,\mathsf{K}(r)))_{r \in \mathbb{N}}$ *is uniformly integrable.*

T.Lewis $(d = 1)$

The proof employs the Bernstein method, the characteristic functions technique, the properties of slowly varying functions and the Lindeberg theorem for arrays.
We use also the following description of the behavior of the partial sums variances.

Lemma

Let $X = \{X_j, j \in \mathbb{Z}^d\}$ *be a wide-sense stationary random field with nonnegative covariance function. If* $\mathrm{K}_X(\cdot) \in \mathcal{L}(\mathbb{N}^d)$ *then*

$$\mathrm{var} S_n \sim \langle n \rangle \, \mathrm{K}_X(n) \quad \text{as} \quad n \to \infty. \quad (4.3)$$

If $\mathrm{var} S_n \sim \langle n \rangle \, L(n)$ *as* $n \to \infty$ *where* $L \in \mathcal{L}(\mathbb{N}^d)$ *then* $L(n) \sim \mathrm{K}_X(n)$, $n \to \infty$.

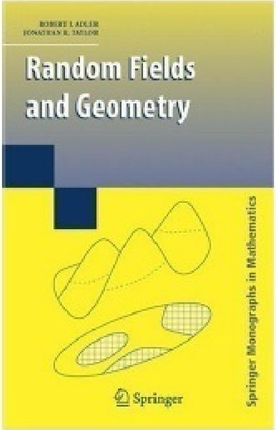
5. Excursion sets

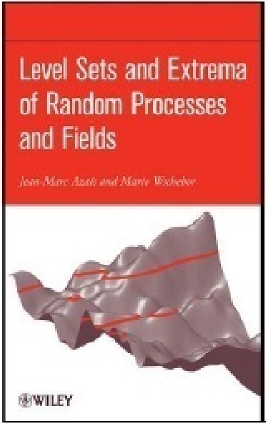
Definition

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function and $T \subset \mathbb{R}^d$ a Lebesgue measurable set. One calls

$$A_u(f, T) = \{t \in T : f(t) \geq u\}$$

the excursion set of f in T over level u.





These books can be viewed as the main sources of information on excursion sets of random fields

Let $X = \{X(t), t \in \mathbb{R}^d\}$ be a real-valued random field defined on $(\Omega, \mathcal{F}, \mathsf{P})$.
Let X be measurable function on $(\mathbb{R}^d \times \Omega, \mathcal{B}(\mathbb{R}^d) \times \mathcal{F})$. Then

$$A_u(X, T) \in \mathcal{B}(\mathbb{R}^d) \times \mathcal{F}.$$

Let $\nu_d(B)$ stand for the Lebesgue measure (volume) of a measurable set $B \subset \mathbb{R}^d$.
Put $\mathbb{I}\{C\}$ for indicator of a set C.

As X is measurable one can claim that, for each $u \in \mathbb{R}$ and every measurable set $T \subset \mathbb{R}^d$, the volume of an excursion set

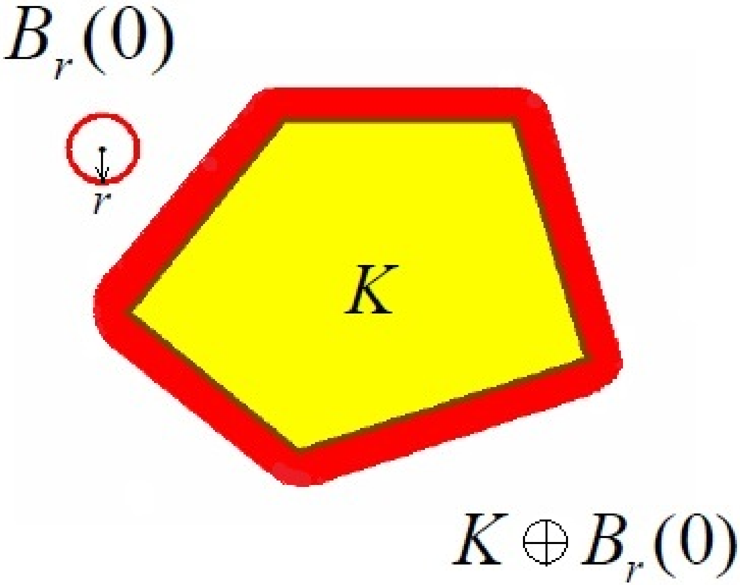
$$\nu_d(A_u(X,T)) = \int_T \mathbb{I}\{X(t) \geq u\} dt$$

is a random variable.

Let ∂B be the border of a set $B \subset \mathbb{R}^d$.

For $A, B \subset \mathbb{R}^d$ consider the Minkowski sum

$$A \bigoplus B = \{x + y : x \in A, y \in B\}$$


$B_r(0)$
r
K
$K \oplus B_r(0)$

The Steiner formula

$$V_d(K \oplus B_r(0)) = \sum_{j=0}^{d} r^{d-j} \kappa_{d-j} V_j(K), \ r \geq 0.$$

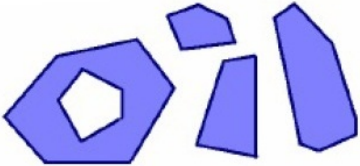
V_j is the intrinsic volume

κ_m *is the volume of the unit ball in* $\mathbb{R}^m$

$$V_d(K) = v_d(K),$$

$$V_{d-1}(K) = \frac{1}{2} S(K),$$

$$V_0(K) = \chi(K).$$



$\mathbb{R}^2, \quad \chi(K) = 4 - 1 = 3.$

Let $\mathcal{K}(d)$ be a class of bodies (bounded convex sets) in $\mathbb{R}^d$.

Theorem (Hadwiger)

Let $f : \mathcal{K}(d) \to \mathbb{R}$ be continuous, rotation invariant, additive function. Then

$$f(C) = \sum_{m=1}^{d} a_m V_m(C), \quad C \in \mathcal{K}(d),$$

for some $a_m \in \mathbb{R}$, $m = 1, \ldots, d$.

Growing sets

For $a=(a_1,\dots,a_d)$ with $a_k>0$, $k=1,\dots,d$, and $j=(j_1,\dots,j_d)\in\mathbb{N}^d$, let

$$\Pi_j(a)=\{x\in\mathbb{R}^d : j_k a_k < x_k \le (j_k+1)a_k,\ k=1,\dots,d.\}$$

Introduce

$$U^-(a)=\bigcup_{j:\Pi_j(a)\subset U}\Pi_j(a),\quad U^+(a)=\bigcup_{j:\Pi_j(a)\cap U\neq\varnothing}\Pi_j(a).$$

Definition

A sequence of sets $U_n \subset \mathbb{R}^d$ *tends to infinity in the Van Hove sense if for any* $a = (a_1, \dots, a_d)$ *with* $a_k > 0$, $k = 1, \dots, d$, *one has*

$$\nu_d(U^-(a))/\nu_d(U^+(a)) \to \infty, \quad n \to \infty.$$

One can prove that if $(W_n)_{n\in\mathbb{N}}$ consists of bounded measurable subsets of $\mathbb{R}^d$ then $W_n \to \infty$ in the Van Hove sense if and only if, for any $\varepsilon > 0$,

$$\nu_d(W_n) \to \infty, \quad \frac{\nu_d(\partial W_n \oplus B_\varepsilon(0))}{\nu_d(W_n)} \to 0,$$

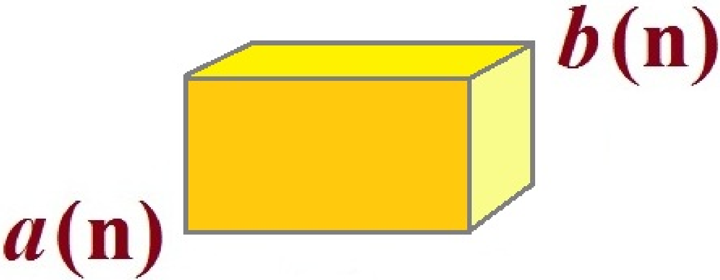
as $n \to \infty$.

Example. Let $W_n = (a(n), b(n)]$ be a parallelepiped, i.e.

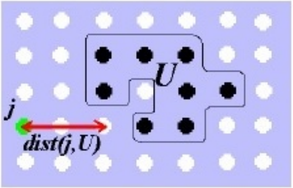
$$W_n = \{x \in \mathbb{R}^d : a_k(n) < x_k \le b_k(n),\ k = 1, \dots, d\}.$$

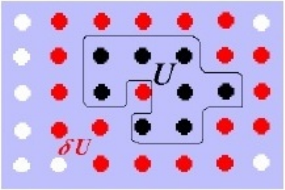
Then $W_n \to \infty$ in the Van Hove sense if and only if, for each $k = 1, \dots, d$,

$$b_k(n) - a_k(n) \to \infty,\ n \to \infty.$$



Let $U \subset \mathbb{Z}^d$, $\delta U = \{j \in \mathbb{Z}^d \setminus U : dist(j, U) = 1\}$.
Here $dist(j, U) = \inf\{\|j - i\|_\infty : i \in U\}$.





Definition

A sequence $(U_n)_{n \in \mathbb{Z}^d}$ of finite subsets of $\mathbb{Z}^d$ tends to infinity in a regular way if

$$\sharp(\delta U_n)/\sharp(U_n) \to 0, \quad n \to \infty.$$

Condition A.

Let a (strictly) stationary random field $X = \{X(t), t \in \mathbb{R}^d\}$ be quasi-associated and $X(0)$ have a bounded density. Assume that covariance function of X is continuous and, for some $\alpha > 3d$, one has

$$|cov(X(0), X(t))| = O(\|t\|^{-\alpha}), \quad \|t\| \to \infty.$$

Here $\|\cdot\|$ is a norm in $\mathbb{R}^d$.

Condition B

Let $X = \{X(t), t \in \mathbb{R}^d\}$ be a stationary Gaussian random field having continuous covariance function. Suppose that, for some $\alpha > d$, the following relation holds

$$|cov(X(0), X(t))| = O(\|t\|^{-\alpha}), \quad \|t\| \to \infty.$$

Theorem (Bulinski, Spodarev, Timmerman)

Let $X = \{X(t), t \in \mathbb{R}^d\}$ be a random field satisfying condition (A) or (B). Then, for any sets $W_n \subset \mathbb{R}^d$ such that $W_n \to \infty$ in the Van Hove sense and each $u \in \mathbb{R}$,

$$\frac{\nu_d(A_u(X,W_n)) - \nu_d(W_n)\mathsf{P}(X(0) \geq u)}{\sqrt{\nu_d(W_n)}} \xrightarrow{law} Z \sim N(0, \sigma^2(u))$$

as $n \to \infty$, here

$$\sigma^2(u) = \int_{\mathbb{R}^d} cov(\mathbb{I}\{X(0) \geq u\}, \mathbb{I}\{X(t) \geq u\})dt \in \mathbb{R}_+.$$

Further generalizations and extensions.

Statistical multidimensional CLT

A.Bulinski, E.Spodarev, F.Timmerman

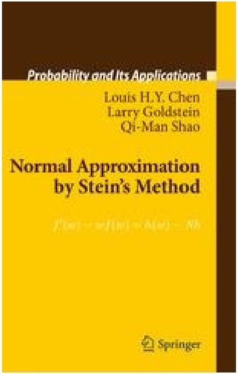
New covariance inequality by V.Demichev for indicators of PA random variables permitting to use for PA fields condition

$$|cov(X(0), X(t))| = O(\|t\|^{-\alpha}), \quad \|t\| \to \infty,$$

with $\alpha > 2d$.

Functional limit theorems established by A.Shashkin (processes indexed by level u).

6. Convergence rates estimates



The Stein equation

$$f'(w) - wf(w) = h(w) - \mathsf{E}h(Z)$$

where $Z \sim N(0,1)$.
For a given ("nice") h, the unique bounded solution of this equation has the form

$$f(w) = e^{w^2/2} \int_{-\infty}^{w} (h(x) - \mathsf{E}(Z))e^{-x^2/2} dx.$$

Let W be a random variable. Then

$$\mathsf{E}f'(W) - Wf(W)) = \mathsf{E}h(W) - \mathsf{E}h(Z)$$

"Leave one out" idea from Stein's original 1972 paper.

Let $X_1, \dots, X_n$ be independent mean zero r.v's with variances $\sigma_1^2, \dots, \sigma_n^2$ such that $\sum_{i=1}^n \sigma_i^2 = 1$. Set

$$W = \sum_{i=1}^n X_i.$$

For some given f, we have

$$\mathrm{E}Wf(W) = \sum_{i=1}^{n} \mathrm{E}X_i f(W) = \mathrm{E}\sum_{i=1}^{n} X_i f(W^{(i)} + X_i)$$

where $W^{(i)} = W - X_i$. If f is differentiable then

$$X_i f(W^{(i)} + X_i) = X_i f(W^{(i)}) + X_i^2 \int_0^1 f'(W^{(i)} + uX_i)\,du.$$

Since $W^{(i)}$ and X_i are independent,

$$\mathrm{E}(Wf(W)) = \mathrm{E}\sum_{i=1}^{n} X_i^2 \int_0^1 f'(W^{(i)} + uX_i)\,du.$$

Further on

$$\mathsf{E}f'(W) = \mathsf{E}\sum_{i=1}^{n}\sigma_i^2 f'(W)$$

$$= \mathsf{E}\sum_{i=1}^{n}\sigma_i^2 f'(W^{(i)}) + \mathsf{E}\sum_{i=1}^{n}\sigma_i^2 (f'(W) - f'(W^{(i)}))$$

$$= \mathsf{E}\sum_{i=1}^{n}X_i^2 f'(W^{(i)}) + \mathsf{E}\sum_{i=1}^{n}\sigma_i^2 (f'(W) - f'(W^{(i)})).$$

Therefore

$$\mathsf{E}(f'(W) - Wf(W))$$

$$= \mathsf{E}\sum_{i=1}^{n} X_i^2 \int_0^1 (f'(W^{(i)} - f'(W^{(i)} + uX_i))du$$

$$+\mathsf{E}\sum_{i=1}^{n} \sigma_i^2 (f'(W) - f'(W^{(i)})).$$

Let f have a bounded f''. Then, for $u \in [0,1]$,

$$|f'(W^{(i)}) - f'(W^{(i)} + uX_i)| \leq |X_i| \|f''\|.$$

Consequently,

$$|\mathsf{E}(f'(W) - Wf(W))|$$
$$\leq \|f''\| \sum_{i=1}^{n} (\mathsf{E}|X_i|^3 + \sigma_i^2 \mathsf{E}|X_i|)$$
$$\leq 2\|f''\| \sum_{i=1}^{n} \mathsf{E}|X_i|^3$$

by the Hölder inequality. Let h be absolutely continuous, then $\|f''\| \leq \|h'\|$.

The Berry-Esseen inequality

Let $X_1, \dots, X_n$ be independent centered random variables such that $\mathsf{E}|X_i|^3 < \infty$, $i = 1, \dots, n$, and $\sum_{i=1}^n \mathsf{E}X_i^2 = 1$. Then

$$\sup_{x\in\mathbb{R}} |\mathsf{P}(W \leq x) - \Phi(x)| \leq C \sum_{i=1}^n \mathsf{E}|X_i|^3$$

where C is a positive constant and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du, \quad x \in \mathbb{R}.$$

$$0.4097 \leq C \leq 7,5900 \quad \text{Esseen' result}$$

General case

$C \leq 0.5591$ I.Tyurin

For i.i.d. random variables

$C \leq 0.4746$ I.Shevtsova

Theorem (Tyurin)

Let $X_1, \ldots, X_n$ be independent centered random variables such that, for some $\delta \in (0,1)$, $\mathsf{E}|X_i|^{2+\delta} < \infty$, $i = 1, \ldots, n$, and $\sum_{i=1}^n \mathsf{E}X_i^2 = 1$. Then

$$\zeta_{2+\delta}(W, Z) \le C_\delta \sum_{i=1}^n \mathsf{E}|X_i|^{2+\delta}$$

where $W = \sum_{i=1}^n X_i$, $Z \sim N(0,1)$ and $C_\delta = 1/(1+\delta)(2+\delta)$. This constant is the best possible.

Here, for random variables U, V and $r > 0$,

$$\zeta_r(U,V) = \sup\{|\mathsf{E}f(U) - \mathsf{E}f(V)| : f \in \mathcal{F}_r\},$$

$$\mathcal{F}_r = \{f : \mathbb{R} \to \mathbb{R} : M_r(f) \leq 1\}.$$

Let $r = k + \alpha$ where $k \in \mathbb{Z}_+$, $\alpha \in (0,1]$. Set

$$M_r(f) = \begin{cases} \sup_{x \neq y} \frac{|f^{(k)}(x) - f^{(k)}(y)|}{|x-y|^{\alpha}} & \text{if } f \in C^{(k)}(\mathbb{R}), \\ +\infty & \text{otherwise.} \end{cases}$$

$C^{(0)}(\mathbb{R}) = C(\mathbb{R})$.