A stochastic characterization of maximal parabolic L^p-regularity

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Problem (MCF-equation)

We consider the mean curvature flow equation

$$\text{(MCF)} \qquad \begin{cases} \partial_t u - \Delta u &= -\sum_{i,j=1}^n \frac{\partial_i u \partial_j u}{1+|\nabla u|^2} \partial_i \partial_j u \quad \textit{in } (0,T) \times \mathbb{R}^n \\ u|_{t=0} &= u_0 \quad \textit{in } \mathbb{R}^n \, . \end{cases}$$

- $x \mapsto (x, u(t,x))$ is the parameterization of a hypersurface in $\mathbb{R}^{n+1}$.
- models the evolution of soap films.
- is a non-linear parabolic PDE.

How can we show the *local* existence and uniqueness of the (strong) solution of the problem?

Problem (MCF-equation)

$$\begin{cases} \partial_t u - \Delta u &= -\sum_{i,j=1}^n \frac{\partial_i u \partial_j u}{1+|\nabla u|^2} \partial_i \partial_j u =: G(u) \quad \text{in } (0,T) \times \mathbb{R}^n \\ u|_{t=0} &= u_0 \quad \text{in } \mathbb{R}^n . \end{cases}$$

We define the solution operator $\mathbb{E} \ni u \mapsto Lu = \begin{pmatrix} \partial_t u - \Delta u \\ u(0) \end{pmatrix} \in \mathbb{F}$. Then u solves (MCF) iff

$$Lu = \begin{pmatrix} G(u) \\ u_0 \end{pmatrix} \Leftrightarrow u = L^{-1} \begin{pmatrix} G(u) \\ u_0 \end{pmatrix}.$$

So: IF L is invertible and $G : \mathbb{E} \to \operatorname{pr}_1 \mathbb{F}$ the equation is reduced to a fixed point problem!

We want to apply the Banach fixed point theorem to $u = L^{-1}\begin{pmatrix} G(u) \\ u_0 \end{pmatrix}$.

We need:

- $\mathbb{E} \ni u \mapsto Lu = \begin{pmatrix} \partial_t u - \Delta u \\ u(0) \end{pmatrix} \in \mathbb{F}$ invertible, that is for all $(f, u_0)^T \in \mathbb{F}$ there is a unique $u \in \mathbb{E}$ with $Lu = (f, u_0)^T$. We *now use*

$$\mathbb{E} := W^{1,p}(0, T; L^p(\mathbb{R}^n)) \cap L^p(0, T; W^{2,p}(\mathbb{R}^n))$$
$$\mathbb{F} := L^p(0, T; L^p(\mathbb{R}^n)) \times \{u(0) : u \in \mathbb{E}\}.$$

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- G maps $\mathbb{E}$ into $\mathrm{pr}_1\,\mathbb{F}$: an important technicality: Sobolev embeddings show: true for $p > n + 2$.

Both points yield: $L^{-1}G$ is a self-mapping, i.e. $L^{-1}\begin{pmatrix} G(u) \\ u_0 \end{pmatrix} : \mathbb{E} \to \mathbb{E}$

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- Strict contraction follows from some basic calculations using $DG(0) = 0$ and the smoothness of G.

The invertibility of $\mathbb{E} \ni u \mapsto Lu = \begin{pmatrix} \partial_t u - \Delta u \\ u(0) \end{pmatrix} \in \mathbb{F}$ is the fundamental concept in our approach.

Definition (Maximal regularity)

X Banach space, $A : D(A) \subset X \to X$ closed operator has *maximal L^p-regularity* if for all $f \in L^p(0, T; X)$ there exists a unique solution

$$u \in \mathbb{E} := W^{1,p}(0, T; X) \cap L^p(0, T; D(A))$$

of the Cauchy problem

$$\begin{cases} \partial_t u(t) + Au(t) & = f(t) \\ u(0) = 0. \end{cases}$$

A closed: $\{(x, Ax) : x \in D(A)\} \subset X \times X$ is closed subspace.

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- Our example (MCF): $X = L^p(\mathbb{R}^n)$
- Defining property holds

$$\text{for } u(0) = 0 \qquad \Rightarrow \qquad \text{for } u(0) = u_0 \in \{w(0) : w \in \mathbb{E}\}.$$

Fourier multiplier theorems: $A = -\Delta$ has maximal L^p-regularity.

Theorem (MCF-equation)

Let $T > 0$, $p \in (n+2, \infty)$. Then there exists $\kappa(T) > 0$ such that

$$(\text{MCF}) \qquad \begin{cases} \partial_t u - \Delta u & = -\sum_{i,j=1}^n \frac{\partial_i u \partial_j u}{1+|\nabla u|^2} \partial_i \partial_j u \quad \textit{in } (0,T) \times \mathbb{R}^n \\ u|_{t=0} & = u_0 \quad \textit{in } \mathbb{R}^n. \end{cases}$$

has a unique solution $u \in \mathbb{E}$ for $\|u_0\|_{W^{2-\frac{2}{p},p}(\mathbb{R}^n)} \leq \kappa$.

How can maximal regularity be characterized?

Theorem (Hilbert space characterization of maximal regularity)

H Hilbert space, A closed operator on H has maximal regularity iff

1. *For every $x \in H$ there exists a unique solution $u_x \in C([0,\infty); H)$ to the integrated problem*

$$u_x(t) = u_0 - \int_0^t A u_x(s)\, ds.$$

2. *The mapping $t \mapsto u_x(t)$ extends to a holomorphic mapping*

$$z \mapsto w_x(z)$$

for all $x \in H$ and z in a sector $\Sigma_\varphi := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \varphi\}$.

- The criteria are easy to check in practice.
- Maximal regularity on Hilbert spaces is too weak for the study of non-linear PDE.

Example

(MCF) needs maximal regularity on $L^p(\mathbb{R}^n)$ for $p > n + 2 > 2$.

Reminder:

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If $X = L^p(\Omega)$ for $p \in (1,\infty)$, do 1) & 2) imply maximal regularity?

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- H. Brézis: formulation of the problem at the beginning of the 80s.
- N. Kalton & G. Lancien: negative answer (2000). They merely showed the existence of a counterexample.
- S.F.: First explicit counterexample (2013).

Where's stochastics? Characterization on $X = L^p(\Omega)$ by L. Weis (2001).

Theorem (Banach space characterization of maximal regularity)

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Theorem (Banach space characterization of maximal regularity)

1. *For every* $x \in H$ *there exists a unique solution* $u_x \in C([0,\infty); X)$ *to the integrated problem*

$$u_x(t) = u_0 - \int_0^t Au_x(s)\, ds.$$

2. *The mapping* $t \mapsto u_x(t)$ *extends to a holomorphic mapping*

$$z \mapsto w_x(z) \qquad (x \in X, z \in \Sigma_\varphi).$$

3. *For i.i.d. Rademacher variables* $r_1, r_2, r_3, \ldots$ $(\mathbb{P}(r_i = \pm 1) = \frac{1}{2})$ *one has*

$$\mathbb{E}\left\| \sum_{k=1}^n r_k w_{x_k}(z_k) \right\| \leq C \cdot \mathbb{E}\left\| \sum_{k=1}^n r_k x_k \right\|$$

for some $C > 0$ *and all* $x_1, \ldots x_n \in X$, $z_1, \ldots, z_n \in \Sigma_\varphi$.

Thank you for your attention!