Trigonometry on groups

Żywilla Fechner | Sept. 2013 | Ulm University

Outline

What is an exponential function

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  only for differentiable functions.
- A non-zero continuous solution of functional equation

$$m(x+y) = m(x)m(y)$$

  with $m(1) := e$.

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- From now on we consider G is an LCA group with a fixed Haar measure k.

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Theorem (Pl. Kannappan, [5])

The function $f\colon G \to \mathbb{R}$ satisfies cosine equation iff there exists an exponential $m\colon G \to \mathbb{C}$ such that
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Theorem (L. Székelyhidi, [7], p. 109)

The pair (g, f) *satisfies Wilson's functional equation iff* $f \equiv 0$ *and* g *arbitrary or* $\exists m_1$ *an exponential,* $\exists a\colon G \to \mathbb{C}$ *additive and* $\exists \alpha \in \mathbb{C}$ *such that* $m_1^2 = 1$ *and for* $x \in G$*:*

$$f(x) = m_1(x), \quad g(x) = (\alpha + a(x))m_1(x), \tag{3}$$

or $\exists m$ *an exponential and constants* $\alpha, \beta \in \mathbb{C}$ *such that* $m^2 \neq 1$ *and for* $x \in G$*:*

$$f(x) = \frac{m(x)+m(-x)}{2}, \quad g(x) = \alpha m(x) + \beta m(-x). \tag{4}$$

Theorem (Wiener, (see Székelyhidi [7], p. 9))

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We present special case of a reasoning from [7].

Example

Let $(\mathbb{Z}_m, +)$ be the additive group of all remainders from division by m equipped with the discrete topology. If $f\colon \mathbb{Z}_m \to \mathbb{C}$ is a bounded function satisfying d'Alembert functional equation for all $x, y \in \mathbb{Z}_m$, then there exists $l \in \mathbb{Z}$ such that

$$f(y) = \cos\left(\frac{2\pi l y}{m}\right), \quad y \in \mathbb{Z}_m.$$

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  - Fix $x, z \in \mathbb{Z}_m$. From (1) applied for $x - z$ instead of x we have

$$f_z(x+y) + f_z(x-y) = 2f_z(x)f(y), \qquad y \in \mathbb{Z}_m.$$

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  - Since a linear combination of solution of (2) is a solution of (2) we obtain

$$g(x+y) + g(x-y) = 2g(x)f(y), \qquad y \in \mathbb{Z}_m$$

for any function $g \in \tau(f)$.

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- There exists an $l \in \{0, 1, \dots, m-1\}$ such that the character

$$\chi(k) = e^{\frac{2\pi ikl}{m}}, \quad k \in \mathbb{Z}_m$$

is an element of $\tau(f)$ (cf. E. Hewitt and K. A. Ross, [4] p. 367), i.e.

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- Dividing by $\chi(x) \neq 0$ we arrive at

$$f(y) = \frac{\chi(y) + \chi(-y)}{2} = \frac{e^{\frac{2\pi ily}{m}} + e^{\frac{-2\pi ily}{m}}}{2} = \cos\left(\frac{2\pi ly}{m}\right),$$

for $y \in \mathbb{Z}_m$.

For a (bounded regular) measure $\mu\colon \mathcal{B}(G) \to \mathbb{C}$ and $y \in G$ we put

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The Fourier transform of a function $f \in L^1(G)$ and a bounded regular Borel measure μ are given by

$$\widehat{f}(\gamma) := \int_G f(x)\check{\gamma}(x)\,dm(x), \quad \widehat{\mu}(\gamma) := \int_G \check{\gamma}(x)\,d\mu(x), \quad \gamma \in \Gamma.$$

where $\check{\gamma}(x) := \gamma(-x)$.

Theorem (Z. Gajda)

Let $\mu\colon \mathcal{B}(G) \to \mathbb{C}$ be a regular bounded measure. Then a function $f \in L^\infty(G)$ which does not vanish m-l.a.e. satisfies equation

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iff there exists a character $\gamma \in \Gamma$ such that

$$f(y) = \widehat{\mu}(\gamma)\gamma(y) + \widehat{\mu}(\check{\gamma})\check{\gamma}(y) \tag{6}$$

$$= \gamma(y)\int_G \gamma(-s)d\mu(s) + \check{\gamma}(y)\int_G \gamma(s)d\mu(s) \tag{7}$$

for all $y \in G$. To get cosine eq. one can take μ given by $\mu(A) := \frac{1}{2}\delta_0(A)$ for $A \in \mathcal{B}(G)$.

We discuss solutions of

$$(f_1 * \mu_y)(x) + (f_2 * (\mu_y)^{-})(x) = g(x)h(y), \quad x, y \in G \tag{8}$$

for $f_1, f_2, g, h\colon G \to \mathbb{C}$ s.t. convolutions are well-defined.

1. Z.F. (2009) **special case** for bounded regular measure μ, $f_1 = f_2 = g$ and h essentially bounded: then there exist a character $\gamma \in \Gamma$ and constants $C_1, C_2 \in \mathbb{C}$ such that g is the form of (6) and

$$h(x) = C_1\gamma(x) - C_2\check{\gamma}(x), \quad x \in G. \tag{9}$$

   Conversely, if $\gamma \in \Gamma$ is a character, $C_1, C_2 \in \mathbb{C}$ are constants and g is given by (6) and h by (9), then (g, g, g, h) fulfills (8).

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2. Z.F. (2011) **special case** for bounded regular measure μ, $f_1 = f_2 = h$ and g essentially bounded.
3. Z.F. & L. Székelyhidi **general case** by means of spectral synthesis.

Spectral synthesis tools

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- *spectral synthesis holds on G* spectral synthesis holds in every proper variety in $\mathcal{C}(G)$.

Spectral synthesis on specific groups

- $(f \in L^1(\mathbb{T}), p \in [1, \infty), g_r(\theta) = \sum_{n\in\mathbb{Z}} r^{|n|}\hat{f}(n)e^{-in\theta}) \Rightarrow$

$$\|f - g_r\|_{L^p} \to 0, \quad r \to 1.$$

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- $f \in \mathcal{C}(\mathbb{T}) \Rightarrow \|f - g_r\|_{\mathcal{C}(\mathbb{T})} \to 0, \quad r \to 1.$
- $f \in L^1(\mathbb{R})$,

$$g_\mu(x) := \int_{\mathbb{R}} e^{-itx}\hat{f}(t)\exp\left(-\frac{t^2\mu}{2}\right)\frac{dt}{2\pi}.$$

If $f \in L^1 \cap L^p$ for $p \in [1,\infty)$, then

$$\|f - g_\mu\|_{L^p} \to 0, \quad \mu \to 0$$

Spectral synthesis tools

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2. If $g(0) = 0$: the general solution; tools: spectral synthesis
3. Remarks on solutions of special case with $f_1 = f_2 = g$ and h for wider class of groups.

Z. F., *A generalization of Gajda equation*,
J. Math. Anal. Appl. 354 (2009) pp. 584-593

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