About the divisors of $a^n + 1$ and their interesting connection to prime numbers

Matthias Heinlein

05.09.2013

Workshop "'Probability, Analysis and Geometry"

Number theoretical basics

Convention

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- If $\gcd(a, c) = 1$, define $\mathrm{ord}_c(a) := \min\{n \mid a^n \equiv 1 \pmod{c}\}$.

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- If $\gcd(a,c) = 1$, define $\mathrm{ord}_c(a) := \min\{n \mid a^n \equiv 1 \pmod{c}\}$. e.g. $\mathrm{ord}_7(2) = 3$, since $2^1 = 2 \not\equiv 1, 2^2 = 4 \not\equiv 1, 2^3 = 8 \equiv 1 \pmod{7}$.

P_a-numbers

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$$a^n + 1$$

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$P_2 = \{1, \cancel{2}, 3, 5, \cancel{7}, 9, 11, 13, 17, 19, 25, 27, 29, 33, ...\}$

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Some criterias

- Divisors of good numbers are good. Multiples of bad numbers are bad.
- If d is an odd prime: d is good $\Leftrightarrow$ $\mathrm{ord}_d(a)$ is even.

Product of good numbers

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Example

$3, 5, 11 \in P_2$.

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The product of two good numbers d and e is good again if and only if $\mathrm{ord}_d(a)$ and $\mathrm{ord}_e(a)$ contain the same power of 2.

Product of good numbers

Example

$3, 5, 11 \in P_2$. $\mathrm{ord}_3(2) = 2$, $\mathrm{ord}_{11}(2) = 10$, both contain 2^1

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$3 \cdot 11 = 33 \in P_2$. $\mathrm{ord}_3(2) = 2$, $\mathrm{ord}_5(2) = 4$, contain different powers of 2.

$3 \cdot 5 = 15 \notin P_2$.

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Woodstone-Visualization

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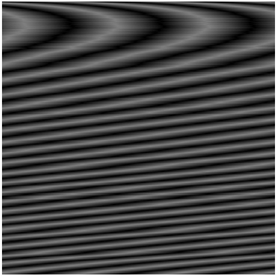

Prime numbers

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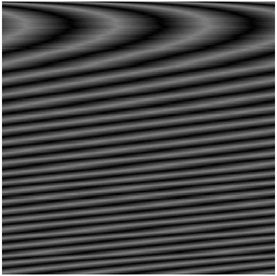

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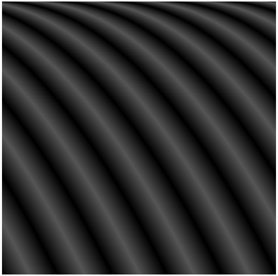

Square numbers

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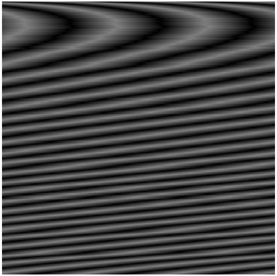

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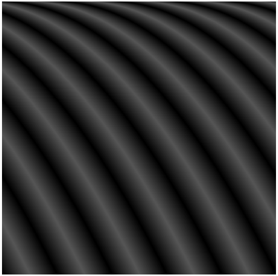

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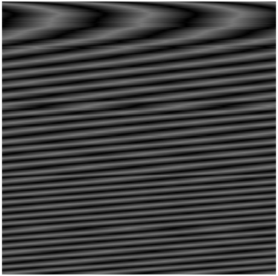

P_2-numbers

Connection with prime numbers

Large gaps in prime numbers

Gaps in prime numbers

2 3 5 7 11 13 17 19 23 29

Large gaps in prime numbers

Gaps in prime numbers

2		3		5		7		11		13		17		19		23		29
	1		2		2		4		2		4		2		4		6	

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One can find abitrarily large gaps between consecutive primes.

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To find a gap of length $\geq n$, define $a := (n+1)!$, so

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One can find abitrarily large gaps between consecutive primes.

Proof.

To find a gap of length $\geq n$, define $a := (n+1)!$, so

$$\begin{aligned} 2 \mid a+2 &\Rightarrow \text{ no prime number} \\ 3 \mid a+3 &\Rightarrow \text{ no prime number} \\ &\dots \\ (n+1) \mid a+(n+1) &\Rightarrow \text{ no prime number} \end{aligned}$$

Large gaps in P_a-numbers

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For every a there are infinitely many bad primes $q_1, q_2, \ldots$.

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To find gaps of length n, find a number x which satisfies the conditions:

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The Chinese Remainder Theorem guarantees a solution for x since $q_1, \ldots, q_n$ are relatively prime. Then $x+1, \ldots, x+n$ are bad numbers.

Famous conjectures about prime numbers

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Definition

A prime twin is a pair (p, q) of two consecutive prime numbers $p < q$ with $q - p = 2$, e.g. $(3, 5), (5, 7), (11, 13), ...)$.

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There are infinitely many prime twins.

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Twin-prime-conjecture

There are infinitely many prime twins.

Goldbach's conjecture (Goldbach, Euler 1742)

Every even number $n \geq 4$ can be expressed as sum of two prime numbers.

Twins in the sets P_a

Twins in the sets P_a

a\n	10	100	1000	10000	100000	1000000	10000000
2	2	13	55	347	2439	17903	140888
3	2	6	35	216	1438	10737	84069
4	0	0	0	0	0	0	0
5	1	7	33	228	1771	13522	109057
6	0	4	24	142	978	7223	56651
7	2	9	39	202	1397	10115	78652
8	2	13	55	347	2439	17903	140888
9	0	0	0	0	0	0	0
10	0	5	27	178	1284	9346	74137
11	1	8	60	317	2279	17229	136758
12	1	5	27	156	1014	7256	55479
13	1	6	30	179	1196	9030	71006
14	2	15	65	404	2757	20449	159570
15	1	5	28	189	1300	9998	79184
16	0	0	0	0	0	0	0
17	2	14	68	420	2984	22590	178247
18	0	3	25	172	1213	8906	69981

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Goldbach's conjecture with P_a-numbers

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In certain cases there is a analogue of the Goldbach's conjecture for P_a-numbers.

Outlook

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Fermat prime numbers

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Fermat prime numbers

Define $Q_d := \{a \mid \ \exists\, n : d \mid a^n + 1\}$ for every $d \geq 2$. Looking for those d with large sets Q_d leads to Fermat prime numbers $2^{2^n} + 1$.

Soli Deo Gloria!

Thank you for your attention!