First-passage percolation on random geometric graphs
and an application to shortest-path trees

Christian Hirsch
jointly with David Neuhäuser, Catherine Gloaguen and Volker Schmidt
First-passage percolation on lattices

Model assumptions and examples

Main results and applications

Open problems
First-passage percolation on lattices

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- $\Rightarrow$ random metric on $\mathbb{Z}^d$
- $\ell(u, v) = \inf_{u = z_0 \sim z_1 \sim \ldots \sim z_n = v} \sum_{i=1}^{n} \tau(\{z_{i-1}, z_i\})$
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- asymptotic behavior of shortest-path lengths $\ell_n = \ell(o, ne_1)$ for large $n$?
- related to notion of tortuosity in materials science
Related models

- *(Howard & Newman, 2001).* first-passage percolation on homogeneous Poisson point process with powers of Euclidean distance as edge-passage times
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  \[ \Rightarrow \text{dependencies in topology} \]
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- this talk. first-passage percolation on random geometric graphs with Euclidean distance as edge-passage times
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  \[ \Rightarrow \text{dependencies in both geometry and topology} \]
First-passage percolation on lattices

Model assumptions and examples

Main results and applications

Open problems
Model assumptions

- \( X \subset \mathbb{R}^d \) stationary, isotropic and \( m \)-dependent point process
- \( N, G \) : family of locally finite sets of points, resp. smooth curves in \( \mathbb{R}^d \)
- \( g : N \rightarrow G \) motion-equivariant, i.e., \( g(\alpha(\varphi)) = \alpha(g(\varphi)) \) for all \( \varphi \in N \) and all rigid motions \( \alpha : \mathbb{R}^d \rightarrow \mathbb{R}^d \)
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- $\left(A_a\right)_{a \geq 1}$ family of events. Say $A_a$ occurs with high probability (whp) if there exists $c_1 > 0$ with $\mathbb{P}(A_a^c) \leq 3 \exp(-a^{c_1})$ for all $a \geq 1$. 
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GSC conditions

- \textit{growth condition}. $\{\nu_1(g(X) \cap Q_1(o)) \leq a\} \cap \{g(X) \cap Q_a(o) \neq \emptyset\}$ occur whp, $Q_a(o) = [-a/2, a/2]^d$. 
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- \textit{stabilization condition}. \( g(X) \cap Q_1(o) = g(X \cap Q_a(o) \cup \psi) \cap Q_1(o) \) for all locally finite \( \psi \subset \mathbb{R}^d \setminus Q_a(o) \) whp.
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- **connectivity condition.** \( g(X) \cap Q_{a/2}(o) \) contained in a connected component of \( g(X) \cap Q_a(o) \) whp.
Examples I

- $X \subset \mathbb{R}^d =$ homogeneous Poisson point process
- Voronoi graph $\text{Vor}(X)$. edge system of tessellation with cells $(C_i)_{i \geq 1}$, where $C_i = \{ y \in \mathbb{R}^d : |y - X_i| \leq |y - X_j| \text{ for all } j \geq 1 \}$

Poisson-Voronoi tessellation
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- Delaunay graph $\text{Del}(X)$. dual graph of Voronoi graph

Poisson-Voronoi tessellation

Poisson-Delaunay tessellation
Examples II

- Creek-crossing graphs $G_n(X)$. For fixed $n \geq 2$ define $G_n(X) = (V, E)$, where $V = X$

- $\{x, y\} \in E$ if $\exists m \leq n, x = x_0, \ldots, x_m = y \in X$ such that $|x_i - x_{i+1}| < |x - y|$ for all $0 \leq i \leq m - 1$
Examples II

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- Framework extendable to further examples, e.g., Johnson-Mehl tessellation, dead leaves model
First-passage percolation on lattices

Model assumptions and examples

Main results and applications

Open problems
Tail bounds for shortest-path lengths

- nearest point on graph: \( q(x) = \arg\min_{y \in G} |x - y| \)
- length of shortest Euclidean path on \( G \): \( \ell(x_1, x_2) = \ell(q(x_1), q(x_2)) \)
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Theorem

- let $X \subset \mathbb{R}^d$ and $G = g(X)$ be as above.
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**Theorem**

- let \( X \subset \mathbb{R}^d \) and \( G = g(X) \) be as above.
- then \( \ell(o, re_1) \leq ur \) whp uniformly over all \( ur \) with \( u \geq u_0 \), i.e.

\[
\exists c_1, c_2 > 0 \text{ with } \mathbb{P}(\ell(o, re_1) \geq ur) \leq c_1 \exp(-ur^{c_2}) \text{ for all } u \geq u_0, r \geq 1
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- proof uses techniques of Deusche & Pisztor (1996)
Tail bounds for shortest-path lengths

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- proof uses techniques of Deuschel & Pisztora (1996)
- see Aldous (2010) for related results in the case \( X=\text{Poisson} \) and \( d = 2 \)
- also true for other ergodic graphs \( G \), e.g. Poisson line tessellation
Applications: boundedness of cells & shape theorem

Corollary

- $G \subset \mathbb{R}^2$ as above
- then a.s. all cells of $G$ are bounded
Applications: boundedness of cells & shape theorem

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**Corollary**

- $G \subset \mathbb{R}^d$ as above
- ⇒ $\exists \mu \geq 1$ such that for all $\varepsilon > 0$ with probability 1:

$$B_{(1-\varepsilon)\mu^{-1}r}(o) \subset B_r^G(o) = \{x \in \mathbb{R}^d : \ell(o, x) \leq r\} \subset B_{(1+\varepsilon)\mu^{-1}r}(o)$$

for all sufficiently large $r \geq 0$

- first-passage metric behaves asymptotically as a scalar multiple of Euclidean metric
Concentration result for moderate deviations

- additional assumptions
  - $X = \text{homogeneous Poisson point process (for simplicity)}$
  - $g(X) = \text{Vor}(X), \text{Del}(X) \text{ or } G_n(X)$

**Theorem**

- for every $\varepsilon > 0$ the events $|\ell(o, re_1) - \mu r| \leq r^{1/2+\varepsilon}$ occur whp
- where $\mu = \lim_{r \to \infty} r^{-1} \mathbb{E}\ell(o, re_1)$
Idea of proof: martingale approach

- fix $\delta > 0$, enumeration $\mathbb{Z}^d = \{z_1, z_2, \ldots\}$
- $(\Omega, \mathcal{F}, \mathbb{P}) = \text{canonical probability space associated with } X$
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- fix $\delta > 0$, enumeration $\mathbb{Z}^d = \{Z_1, Z_2, \ldots \}$
- $(\Omega, \mathcal{F}, \mathbb{P}) = \text{canonical probability space associated with } X$
- consider filtration $\left( \mathcal{F}_k^{(r)} \right)_{k \geq 1}$ of $\mathcal{F}$ and martingale $\left( M_k^{(r)} \right)_{k \geq 1}$, where

$$
\mathcal{F}_k^{(r)} = \sigma \left( X \cap \bigcup_{i=1}^k Q_{r\delta} \left( r\delta z_i \right) \right) \quad \text{and} \quad M_k^{(r)} = \mathbb{E} \left( \ell(o, re_1) \mid \mathcal{F}_k^{(r)} \right)
$$

- apply suitable martingale concentration result (Kesten, 1993)
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  \]
- apply suitable martingale concentration result (Kesten, 1993)
Concentration result for geodesics

- \( \rho_r \subset G \): shortest path from \( q(o) \) to \( q(re_1) \)

**Corollary**

\( \rho_r \subset [o, re_1] \oplus Q_{r^{3/4+\varepsilon}}(o) \) occurs whp for all \( \varepsilon > 0 \)
Shortest-path tree

- consider the graph $G^*$ obtained from $G$ by placing the origin at random on the edge set $G^{(1)}$
- formally, $G^*$ is the Palm version of $G$
Shortest-path tree

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- consider distance peaks
  \[ M = \{ x \in G^* \mid \text{shortest path from } x \text{ to } o \text{ not unique} \} \]
- define the shortest-path tree $T = G^* \setminus M$ by elimination of distance peaks
Shortest-path tree

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- consider distance peaks $M = \{ x \in G^* \mid \text{shortest path from } x \text{ to } o \text{ not unique} \}$
- define the shortest-path tree $T = G^* \setminus M$ by elimination of distance peaks
- then, for any $x \in T$: the path from $x$ to $o$ in $T$ equals shortest path from $x$ to $o$ in $G^*$
Poisson-Delaunay graph (cutout)
Shortest-path tree (cutout)
Competition interface

- (Howard & Newman, 2001). concentration result for $\rho_r \Rightarrow$ existence of asymptotic directions (AD) for all semi-infinite paths $\gamma \subset T$
- i.e. for $x_k/|x_k| \to \theta \in S^{d-1}$, where $\gamma = \langle x_1, x_2, \ldots \rangle$
Competition interface

- \((\text{Howard} \ & \ \text{Newman}, \ 2001)\). concentration result for \(\rho_r \Rightarrow \) existence of \textit{asymptotic directions (AD)} for all semi-infinite paths \(\gamma \subset T\)
- i.e. for \(x_k/ |x_k| \to \theta \in S^{d-1}\), where \(\gamma = \langle x_1, x_2, \ldots \rangle\)
- \(d = 2\)
- write \(T_1, T_2\) for subtrees at \(o\) (observe \(\text{deg}_T(o) = 2\) a.s.)
- define \textit{competition interface} \(I = \overline{T_1} \cap \overline{T_2}\)
Competition interface

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- $d = 2$
- write $T_1, T_2$ for subtrees at $o$ (observe $\text{deg}_T(o) = 2$ a.s.)
- define competition interface $I = \overline{T_1} \cap \overline{T_2}$
- if $\nu_1(T_1) = \nu_1(T_2) = \infty$, then $I = I_1 \cup I_2$ where $I_1$ and $I_2$ admit AD
Poisson-Delaunay graph (cutout)
Competition interface
Poisson-Delaunay graph (cutout)
Competition interface: pathological realization
First-passage percolation on lattices

Model assumptions and examples

Main results and applications

Open problems
Open problems

- $d = 2$
  - two unbounded half-trees occur with positive probability
  - non-existence of bi-infinite geodesics with fixed asymptotic directions
  - coalescence of semi-infinite geodesics with same fixed asymptotic directions
Thank you for your attention!
D. J. Aldous.  
Which connected spatial networks on random points have linear route-lengths?  

J. Deuschel and A. Pisztora.  
Surface order large deviations for high-density percolation.  

First-passage percolation on random geometric graphs and an application to shortest-path trees.  

C. Hirsch, D. Neuhäuser, and V. Schmidt.  
Moderate deviations for shortest-path lengths on random geometric graphs. (Working Paper).  
2013.

C. Howard and C. Newman.  
Euclidean models of first-passage percolation.  

C. Howard and C. Newman.  
Geodesics and spanning trees for Euclidean first-passage percolation.  

H. Kesten.  
On the speed of convergence in first-passage percolation.  
Bivariate distribution of backbone lengths

\[ Z^{(i)}(\lambda_H) = \sup_{x \in S_{\lambda_H} \cap T_i} \ell(o, x), \ i \in \{1, 2\} \]

**Theorem**

\[ X \subset \mathbb{R}^2. \text{ homogeneous Poisson point process with intensity } \gamma \]
Bivariate distribution of backbone lengths

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**Theorem**

- \( X \subset \mathbb{R}^2 \). homogeneous Poisson point process with intensity \( \gamma \)
- then, \( \exists \) random vector \((Z^{(1)}, Z^{(2)})\)

\[
\left( Z^{(1)}(\lambda_H) \sqrt{\lambda_H}, Z^{(2)}(\lambda_H) \sqrt{\lambda_H} \right) \xrightarrow{d} \left( Z^{(1)}, Z^{(2)} \right),
\]

as \( \lambda_H \to 0 \), *conditioned on simultaneous unboundedness of \( T_1 \) and \( T_2 \)
Bivariate distribution of backbone lengths (II)

- explicit interpretation of \((Z^{(1)}, Z^{(2)})\)
Bivariate distribution of backbone lengths (II)

- explicit interpretation of \((Z^{(1)}, Z^{(2)})\)
- generate typical Poisson-Voronoi cell
Bivariate distribution of backbone lengths (II)

- explicit interpretation of $(Z^{(1)}, Z^{(2)})$
- generate independent, isotropic sector
Bivariate distribution of backbone lengths (II)

- explicit interpretation of \((Z^{(1)}, Z^{(2)})\)
- find most distant point in each part and put \(Z^{(i)} = \xi \cdot \zeta^{(i)}, i = 1, 2\)
A moderate deviation result

**Theorem (Alexander, Kesten, 1993)**

- if $\mathbb{P}(\tau = 0) < p_c(\mathbb{Z}^d)$ and $\mathbb{E}(\tau^2) < \infty$, then $\exists C_1, C_2, C_3, C_4, C_5 > 0$ such that $\forall n \geq 1$
A moderate deviation result

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  - $C_1 n^{-2} \leq \frac{1}{n} \mathbb{E} \ell_n - \mu \leq C_2 n^{-1/2} \log(n)$
  
  - $P(|\ell_n - \mathbb{E} \ell_n| \geq \sqrt{n}x) \leq C_3 \exp(-C_4x)$ for all $x \leq C_5 n$
  
  - $P(\ell_n - n\mu \leq -\sqrt{n}x) \leq C_3 \exp(-C_4x)$ for all $x \leq C_5 n$
  
  - $P(\ell_n - n\mu \geq 2C_2 n^{1/2} \log(n) + xn^{1/2}) \leq C_3 \exp(-C_4x)$ for all $x \leq C_5 n$
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- variance of $\ell_n$ conjectured to be of order $n^{2/3}$
- best upper-bound by (Benjamini, Kalai & Schramm, 2003) is $O(n/\log(n))$
A moderate deviation result

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- Variance of \( \ell_n \) conjectured to be of order \( n^{2/3} \)
- Best upper-bound by (Benjamini, Kalai & Schramm, 2003) is \( O(n/\log(n)) \)
- No CLT known for \( \ell_n \); limiting distribution conjectured to be of Tracy-Widom type
Main result

Theorem

- $X \subseteq \mathbb{R}^d$ homogeneous Poisson point process
- let $G = \text{Vor}(X)$ or $G = \text{Del}(X)$
Main result

Theorem

\( X \subset \mathbb{R}^d \) homogeneous Poisson point process

let \( G = \text{Vor}(X) \) or \( G = \text{Del}(X) \)

then for all \( \varepsilon > 0 \) the events \( |\ell_r - \mu r| \leq r^{1/2+\varepsilon} \) occur whp

where \( \mu = \lim_{r \to \infty} r^{-1} E \ell_r \)
Concentration Result

Lemma (Kesten, 1993)

\[(\Omega, \mathcal{F}, \mathbb{P}) \text{ probability space with filtration } (\mathcal{F}_k)_{k \geq 0}\]
Concentration Result

Lemma (Kesten, 1993)

- $(\Omega, \mathcal{F}, \mathbb{P})$ probability space with filtration $(\mathcal{F}_k)_{k \geq 0}$
- $\Delta_k = M_k - M_{k-1}$ for some $(\mathcal{F}_k)_{k \geq 0}$-martingale $(M_k)_{k \geq 0}$, $M_0 = 0$. 
Concentration Result

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- $(U_k)_{k \geq 1}$ sequence of $\mathcal{F}$-measurable rvs with $\mathbb{E}(\Delta_k^2 | \mathcal{F}_{k-1}) \leq \mathbb{E}(U_k | \mathcal{F}_{k-1})$ a.s.
- $S = \sum_{k=1}^{\infty} U_k$ and assume $\exists C'_1 > 0$, $0 < \gamma \leq 1$, $c \geq 1$ and $x_0 \geq c^2$ with $|\Delta_k| \leq c$ a.s. and $\mathbb{P}(S > x) \leq C'_1 \exp(-x^{\gamma})$ for all $x \geq x_0$
Concentration Result

**Lemma (Kesten, 1993)**

- $(\Omega, \mathcal{F}, P)$ probability space with filtration $(\mathcal{F}_k)_{k \geq 0}$
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- then $\lim_{k \to \infty} M_k = M < \infty \ a.s. \ and \ \exists C_2 = C_2(C'_1, \gamma) \ and \ C_3 = C_3(\gamma) > 0$ such that
  $$\mathbb{P}(|M| \geq x\sqrt{x_0}) \leq C_2 \exp(-C_3 x) \ for \ all \ x \leq x_0^{\gamma}$$
Application to shortest-path problem

- **first approach.** \( \mathcal{F}_k^{(n)} = \sigma \left( X \cap \bigcup_{i=1}^k Q_{n^\delta} \left( n^\delta z_i \right) \right) \) and \( M_k^{(n)} = \mathbb{E} \left( \ell_n \mid \mathcal{F}_k^{(n)} \right) - \mathbb{E} \ell_n \)

- however, \( \Delta_k^{(n)} \) not bounded by a constant uniformly in \( k \)!
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- show \( \ell_g(X)(o, ne_1) = \ell_g(X(n^\delta))(o, ne_1) \) whp
Straightness

- for $x \in T$ write $T_x \subseteq T$ for descendant tree at $x$
- consider cone $C(x, \delta) = \{ y \in \mathbb{R}^d : |\angle(x, y)| \leq \delta \}$
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- existence of *asymptotic directions (AD)* for all semi-infinite paths $\gamma \subset T$
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- $\forall \theta \in S^{d-1}$ there exists a semi-infinite path in $\gamma \subset T$ with AD $\theta$
- the set of those $\theta \in S^{d-1}$ for which there exists more than one semi-infinite path in $T$ with AD $\theta$ is dense in $S^{d-1}$
Outlook

- application in telecommunication networks
  - asymptotic description for joint distribution for the lengths of longest branches in both of the two subtrees at the origin
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- enlarge class of examples for which MD-result holds (e.g. dead-leaves model, Johnson-Mehl tessellations, creek-crossing graphs)
- non-existence of bi-infinite geodesics \((d = 2)\)
- coalescence of semi-infinite geodesics with same asymptotic directions \((d = 2)\)
Further developments

- FPP on random geometric graphs: studied by Vahidi-Asl & Wierman (1990), Baccelli, Błaszczyszyn & Haji-Mirsadeghi (2011), Pimentel (2011)
- mostly iid edge passage times (e.g. hopcounts)
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- mostly iid edge passage times (e.g. hopcounts)
- example: Delaunay graph/relative neighborhood graph on a homogeneous PPP in dimension $d = 2$
First step: block construction

- follow approach proposed by Deuschel & Pisztora (1996)
- consider discretization of $\mathbb{R}^d$ into cubes of side length $L > 0$
- define site percolation process of good cubes with the properties
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$\Rightarrow$ stochastically dominates supercritical Bernoulli percolation process
Construction of global & local routes

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- **final step.** construction of local routes from \( o, re_1 \) to global path
- choose any path leaving bad clusters close to \( o \)
- whp these clusters are not too large and length of local path length bounded from above by total edge length inside certain bad clusters
Stochastic Subscriber Network

- main roads

simulated main roads
Stochastic Subscriber Network

- main roads
- side streets

simulated side streets
Stochastic Subscriber Network

- main roads
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- network components:
  - higher-level components (green)
  - lower-level components (blue)

network components along roads
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- serving zones of higher-level components
Shortest-path trees

- consider segment system $S^*_H$ of a typical serving zone
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Shortest-path trees

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- goal. asymptotic characteristics of *backbone lengths* in unboundedly dense networks?
  - asymptotic marginal distribution of longest-branch length?
  - asymptotic joint distribution of longest-branch lengths in both subtrees?

![Diagram of shortest-path trees](image)
Applications: boundedness of cells & shape theorem

Corollary

- $G \subset \mathbb{R}^2$ as above
- then a.s. all cells of $G$ are bounded
Applications: boundedness of cells & shape theorem

Corollary

- $G \subset \mathbb{R}^2$ as above
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idea: use theorem to construct closed curve of edges around the origin

consider annulus of squares of side length $\sqrt{r}$ at distance $r^2$ from $o$ and distance $r$ from each other

by the thm: points in neighboring squares can be connected by a path far away from $o$ whp

$\Rightarrow$ claim follows from Borel-Cantelli lemma
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serving zones
Description of the model

- $G = \text{random geometric graph in } \mathbb{R}^2 \text{ as above}$
- $G^* = \text{Palm version of } G$
  - informally: shifting $o$ to random location on the edge set of $G$
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- $Z(\lambda) = \sup_{x \in S^*_\lambda} \ell(o, x)$
Asymptotic result

Theorem

- $\Xi = \text{typical Poisson-Voronoi cell based on homogeneous Poisson point process with intensity } \mathbb{E}_{\nu_1}(G \cap Q_1(o))$
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- distribution of $R$ explicitly known, see Calka 2002
Idea of proof

- by the shape theorem: length of longest branch $\approx \xi \cdot$ Euclidean distance from most distant point in the serving zone to the origin

- furthermore: scaled typical serving zone converges asymptotically to typical Poisson-Voronoi cell
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- tail bound can be refined for probabilities of moderate deviations of shortest-path lengths

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- based on martingale concentration inequality due to Kesten (1993)
Asymptotic joint distribution of backbone lengths

- so far, asymptotic (one-dimensional) distribution of longest branch in shortest-path tree (backbone)
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