First-passage percolation on random geometric graphs and an application to shortest-path trees

Christian Hirsch
jointly with David Neuhäuser, Catherine Gloaguen and Volker Schmidt

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Model assumptions and examples

Main results and applications

Open problems

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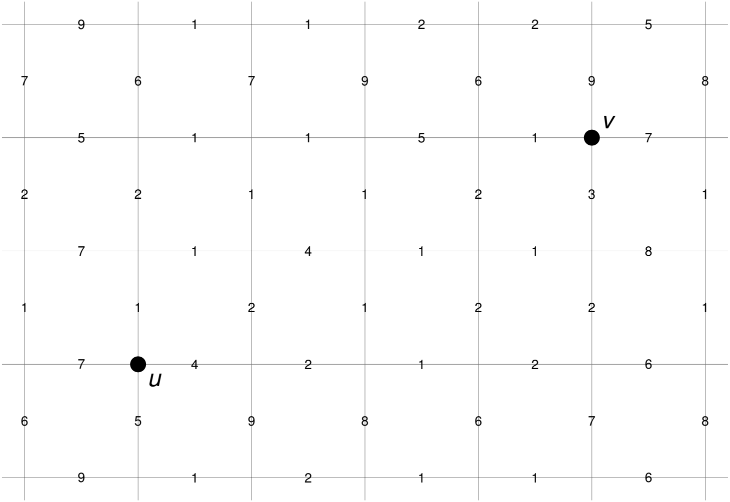
- asymptotic behavior of *shortest-path lengths* $\ell_n = \ell(o, ne_1)$ for large n?
- related to notion of *tortuosity* in materials science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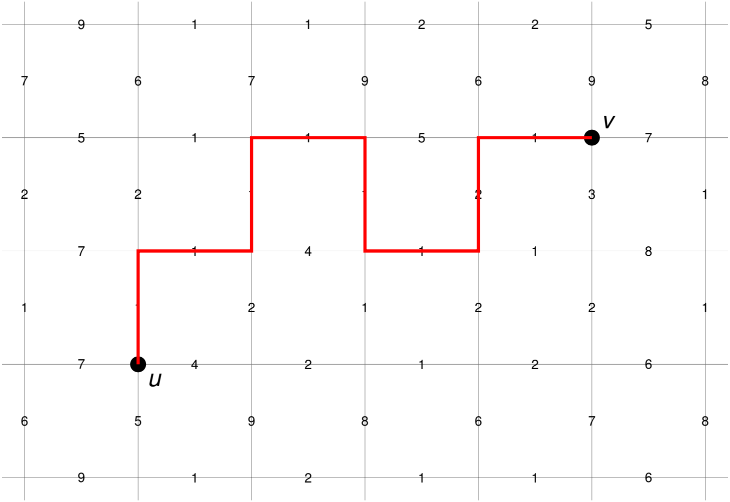
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Model assumptions

- $X \subset \mathbb{R}^d$ stationary, isotropic and *m*-dependent point process
- $\mathbb{N}, \mathbb{G}$: family of locally finite sets of points, resp. smooth curves in $\mathbb{R}^d$
- $g : \mathbb{N} \to \mathbb{G}$ motion-equivariant, i.e., $g(\alpha(\varphi)) = \alpha(g(\varphi))$ for all $\varphi \in \mathbb{N}$ and all rigid motions $\alpha : \mathbb{R}^d \to \mathbb{R}^d$

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- $(A_a)_{a \geq 1}$ family of events. Say A_a occurs *with high probability (whp)* if there exists $c_1 > 0$ with $\mathbb{P}\left(A_a^c\right) \leq 3 \exp\left(-a^{c_1}\right)$ for all $a \geq 1$

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GSC conditions

- *growth condition.* $\{\nu_1(g(X) \cap Q_1(o)) \leq a\} \cap \{g(X) \cap Q_a(o) \neq \emptyset\}$ *occur whp,* $Q_a(o) = [-a/2, a/2]^d$.

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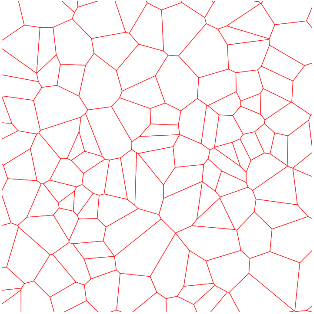
- *connectivity condition.* $g(X) \cap Q_{a/2}(o)$ *contained in a connected component of* $g(X) \cap Q_a(o)$ *whp.*

Examples I

- $X \subset \mathbb{R}^d$ = homogeneous Poisson point process
- Voronoi graph $Vor(X)$. edge system of tessellation with cells $(C_i)_{i \geq 1}$, where $C_i = \{y \in \mathbb{R}^d : |y - X_i| \leq \left|y - X_j\right| \text{ for all } j \geq 1\}$



Poisson-Voronoi tessellation

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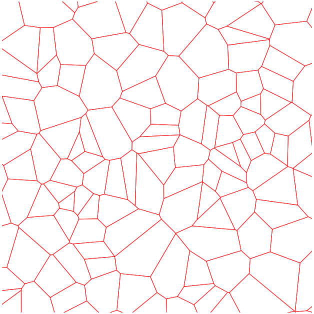
- Delaunay graph $Del(X)$. dual graph of Voronoi graph



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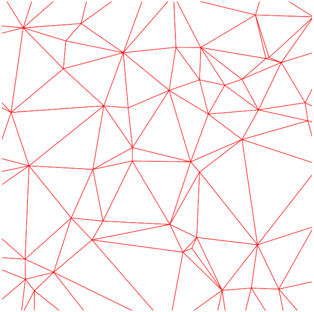

Poisson-Delaunay tessellation

Examples II

- Creek-crossing graphs $G_n(X)$. for fixed $n \geq 2$ define $G_n(X) = (V, E)$, where $V = X$
- $\{x, y\} \in E$ if $\nexists m \leq n, x = x_0, \ldots, x_m = y \in X$ such that $|x_i - x_{i+1}| < |x - y|$ for all $0 \leq i \leq m - 1$

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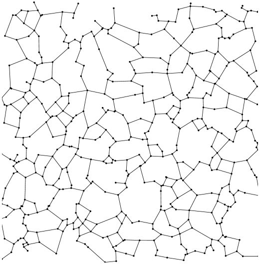
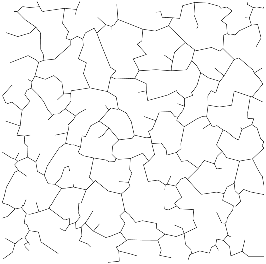


$n = 2$



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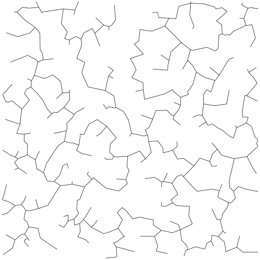

$n = 10$

- framework extendable to further examples, e.g., Johnson-Mehl tessellation, dead leaves model

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Tail bounds for shortest-path lengths

- nearest point on graph: $q(x) = \mathrm{argmin}_{y \in G} |x - y|$
- length of shortest Euclidean path on G: $\ell(x_1, x_2) = \ell(q(x_1), q(x_2))$

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- *then $\ell(o, re_1) \le ur$ whp uniformly over all ur with $u \ge u_0$, i.e.*

$$\exists c_1, c_2 > 0 \text{ with } \mathbb{P}\left(\ell(o, re_1) \ge ur\right) \le c_1 \exp\left(-(ur)^{c_2}\right) \text{ for all } u \ge u_0, r \ge 1$$

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- see Aldous (2010) for related results in the case X=Poisson and $d = 2$
- also true for other ergodic graphs G, e.g. Poisson line tessellation

Applications: boundedness of cells & shape theorem

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- *$G \subset \mathbb{R}^d$ as above*
- *$\Rightarrow \exists \mu \geq 1$ such that for all $\varepsilon > 0$ with probability 1:*

$$B_{(1-\varepsilon)\mu^{-1}r}(o) \subset B_r^G(o) = \{x \in \mathbb{R}^d : \ell(o,x) \leq r\} \subset B_{(1+\varepsilon)\mu^{-1}r}(o)$$

for all sufficiently large $r \geq 0$

- first-passage metric behaves asymptotically as a scalar multiple of Euclidean metric

Concentration result for moderate deviations

- additional assumptions
  - $X =$ homogeneous Poisson point process (for simplicity)
  - $g(X) = Vor(X)$, $Del(X)$ or $G_n(X)$

Theorem

- *for every* $\varepsilon > 0$ *the events* $|\ell(o, re_1) - \mu r| \leq r^{1/2+\varepsilon}$ *occur whp*
- *where* $\mu = \lim_{r\to\infty} r^{-1}\mathbb{E}\ell(o, re_1)$

Idea of proof: martingale approach

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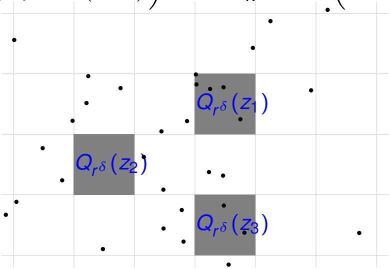

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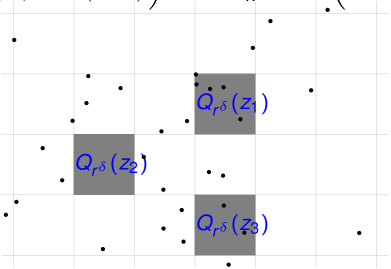


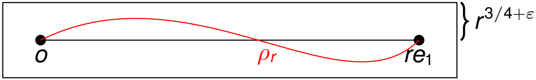
- apply suitable martingale concentration result (Kesten, 1993)

Concentration result for geodesics

- $\rho_r \subset G$: shortest path from $q(o)$ to $q(re_1)$

Corollary

$\rho_r \subset [o, re_1] \oplus Q_{r^{3/4+\varepsilon}}(o)$ *occurs whp for all* $\varepsilon > 0$



Shortest-path tree

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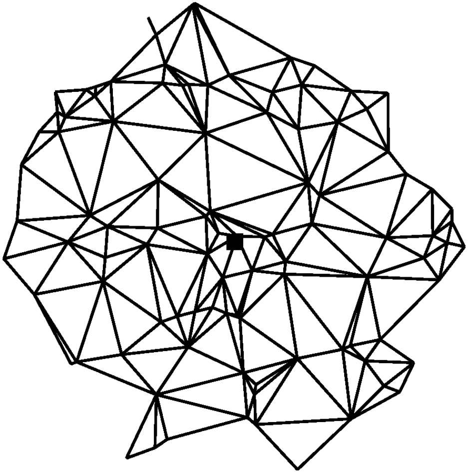
- define the *shortest-path tree* $T = G^* \setminus M$ by elimination of distance peaks
- then, for any $x \in T$: the path from x to o in T equals shortest path from x to o in G^*

Poisson-Delaunay graph (cutout)



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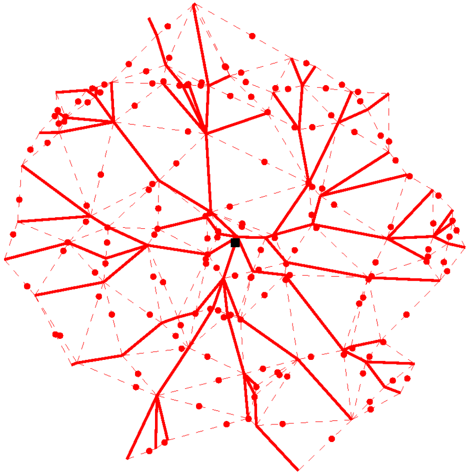

Competition interface

- *(Howard & Newman, 2001).* concentration result for $\rho_r \Rightarrow$ existence of *asymptotic directions (AD)* for all semi-infinite paths $\gamma \subset T$
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- $d = 2$
- write T_1, T_2 for subtrees at o (observe $deg_T(o) = 2$ a.s.)
- define *competition interface* $I = \overline{T_1} \cap \overline{T_2}$

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- if $\nu_1(T_1) = \nu_1(T_2) = \infty$, then $I = I_1 \dot\cup I_2$ where I_1 and I_2 admit AD

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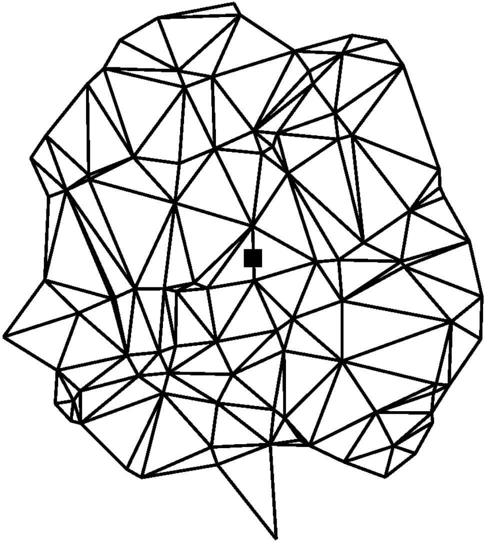

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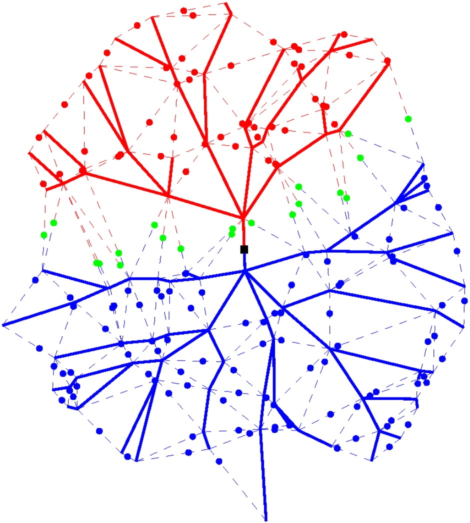

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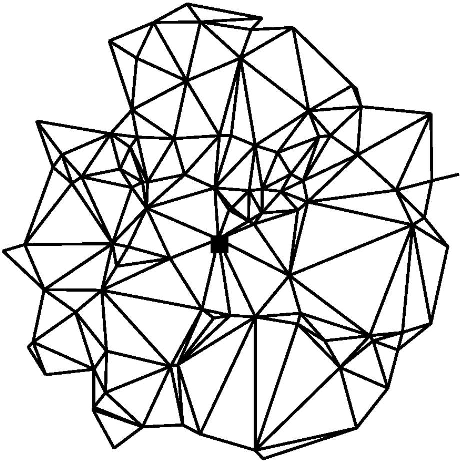
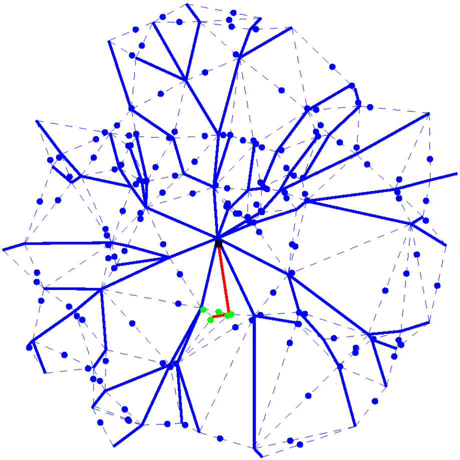


Competition interface: pathological realization



Open problems

- $d = 2$
  - two unbounded half-trees occur with positive probability
  - non-existence of bi-infinite geodesics with fixed asymptotic directions
  - coalescence of semi-infinite geodesics with same fixed asymptotic directions

Thank you for your attention!

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Bivariate distribution of backbone lengths

- $Z^{(i)}(\lambda_H) = \sup_{x \in S^*_{\lambda_H} \cap T_i} \ell(o, x)$, $i \in \{1, 2\}$

Theorem

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Theorem

- *$X \subset \mathbb{R}^2$. homogeneous Poisson point process with intensity γ*
- *then, $\exists$ random vector $\left(Z^{(1)}, Z^{(2)}\right)$*

$$\left(Z^{(1)}(\lambda_H)\sqrt{\lambda_H}, Z^{(2)}(\lambda_H)\sqrt{\lambda_H}\right) \xrightarrow{\mathrm{d}} \left(Z^{(1)}, Z^{(2)}\right),$$

as $\lambda_H \to 0$, conditioned on simultaneous unboundedness of T_1 and T_2

Bivariate distribution of backbone lengths (II)

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- explicit interpretation of $\left(Z^{(1)}, Z^{(2)}\right)$
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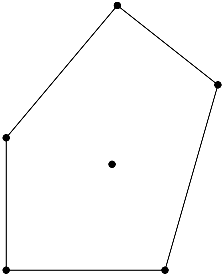
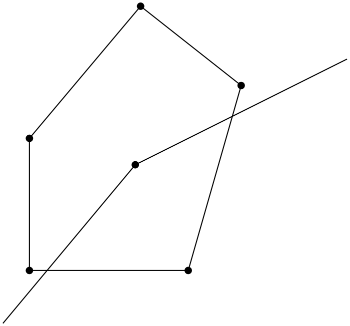


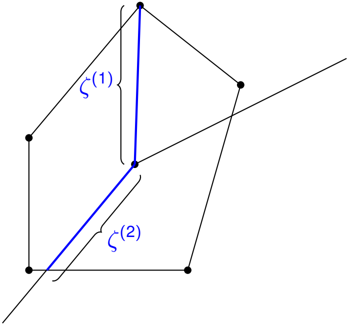
Bivariate distribution of backbone lengths (II)

- explicit interpretation of $\left(Z^{(1)}, Z^{(2)}\right)$
- generate independent, isotropic sector



Bivariate distribution of backbone lengths (II)

- explicit interpretation of $\left(Z^{(1)}, Z^{(2)}\right)$
- find most distant point in each part and put $Z^{(i)} = \xi \cdot \zeta^{(i)}$, $i = 1, 2$



A moderate deviation result

Theorem (Alexander, Kesten, 1993)

- *if* $\mathbb{P}(\tau = 0) < p_c(\mathbb{Z}^d)$ *and* $\mathbb{E}\left(\tau^2\right) < \infty$, *then* $\exists C_1, C_2, C_3, C_4, C_5 > 0$ *such that* $\forall n \geq 1$

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- no CLT known for ℓ_n; limiting distribution conjectured to be of Tracy-Widom type

Main result

Theorem

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- *where $\mu = \lim_{r\to\infty} r^{-1}\mathbb{E}\ell_r$*

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Lemma (Kesten, 1993)

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- *then* $\lim_{k\to\infty} M_k = M < \infty$ *a.s. and* $\exists C_2 = C_2(C_1', \gamma)$ *and* $C_3 = C_3(\gamma) > 0$ *such that*

$$\mathbb{P}\left(|M| \geq x\sqrt{x_0}\right) \leq C_2 \exp(-C_3 x) \qquad \text{for all } x \leq x_0^\gamma$$

Application to shortest-path problem

- *first approach.* $\mathcal{F}_k^{(n)} = \sigma\left(X \cap \bigcup_{i=1}^{k} Q_{n^\delta}\left(n^\delta z_i\right)\right)$ and $M_k^{(n)} = \mathbb{E}\left(\ell_n \mid \mathcal{F}_k^{(n)}\right) - \mathbb{E}\ell_n$
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- show $\ell_{g(X)}(o, ne_1) = \ell_{g(X(n^\delta))}(o, ne_1)$ whp

Straightness

- for $x \in T$ write $T_x \subset T$ for descendant tree at x
- consider cone $C(x, \delta) = \{y \in \mathbb{R}^d : |\angle(x, y)| \leq \delta\}$

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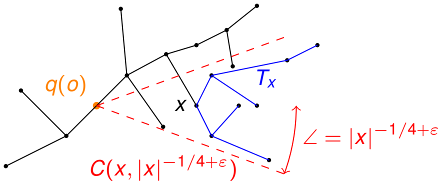
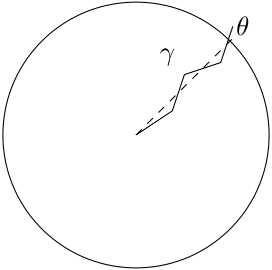


Implications of straightness

- existence of *asymptotic directions (AD)* for all semi-infinite paths $\gamma \subset T$
- i.e. for $\gamma = \langle x_1, x_2, \ldots \rangle$ we have $x_k / |x_k| \to \theta \in S^{d-1}$



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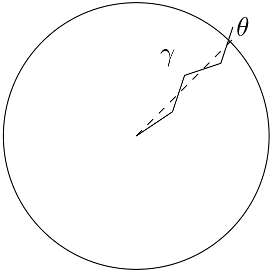

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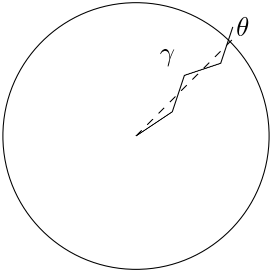
- the set of those $\theta \in S^{d-1}$ for which there exists more than one semi-infinite path in T with AD θ is dense in S^{d-1}



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- enlarge class of examples for which MD-result holds (e.g. dead-leaves model, Johnson-Mehl tessellations, creek-crossing graphs)
- non-existence of bi-infinite geodesics ($d = 2$)
- coalescence of semi-infinite geodesics with same asymptotic directions ($d = 2$)

Further developments

- FPP on random geometric graphs: studied by Vahidi-Asl & Wierman (1990), Baccelli, Błaszczyszyn & Haji-Mirsadeghi (2011), Pimentel (2011)
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- example: Delaunay graph/relative neighborhood graph on a homogeneous PPP in dimension $d = 2$

First step: block construction

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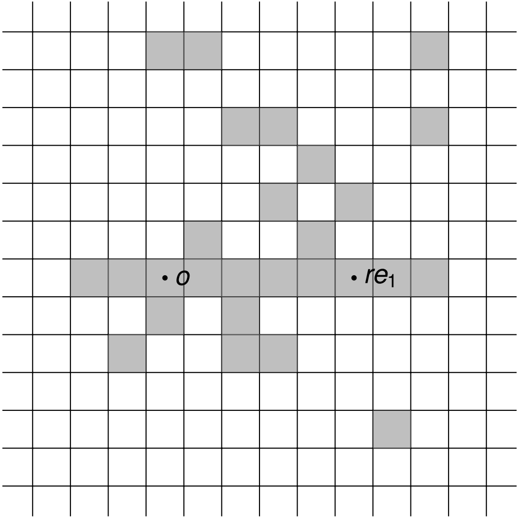

o
re_1

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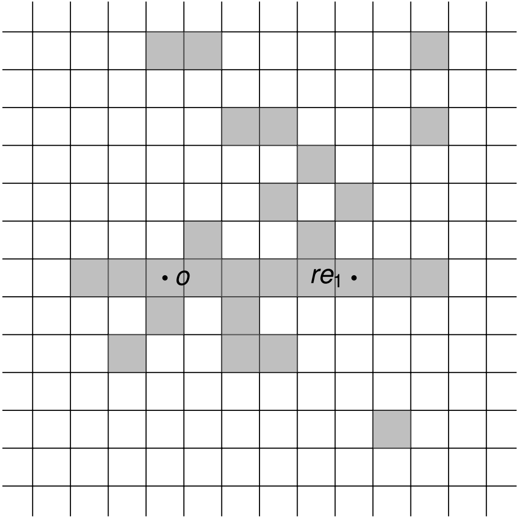
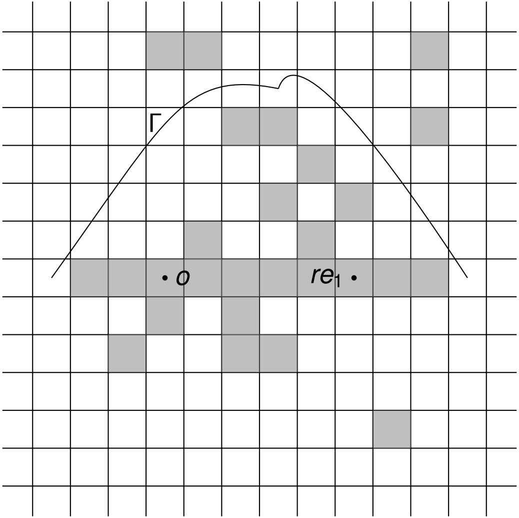
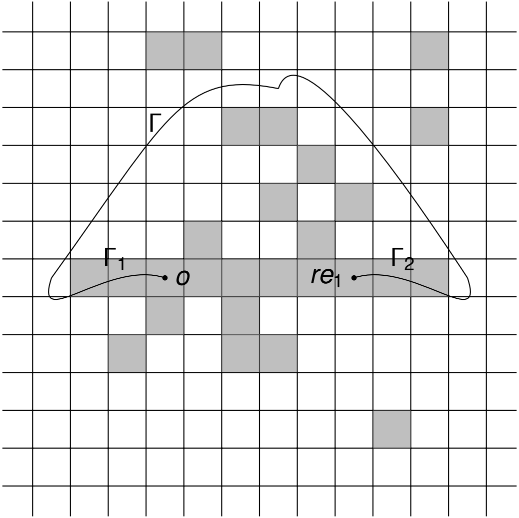
- choose any path leaving bad clusters close to o
- whp these clusters are not too large and length of local path length bounded from above by total edge length inside certain bad clusters


o
re1

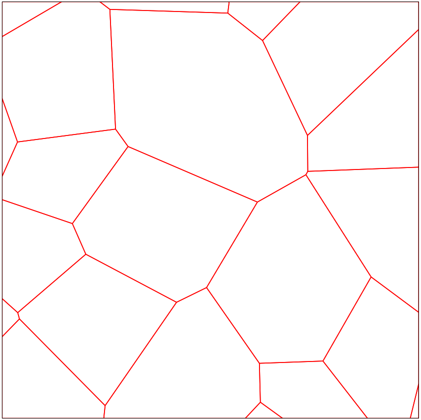


Γ
o
re_1



Γ
Γ_1
o
re_1
Γ_2

Stochastic Subscriber Network



simulated main roads

- main roads

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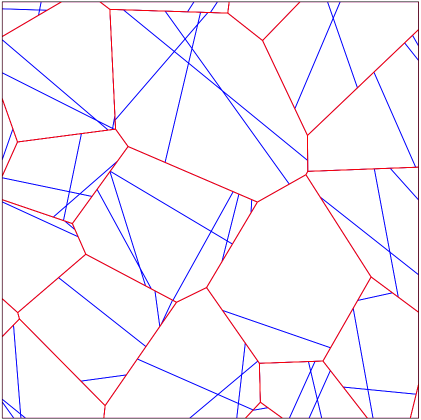

simulated side streets

- main roads
- side streets

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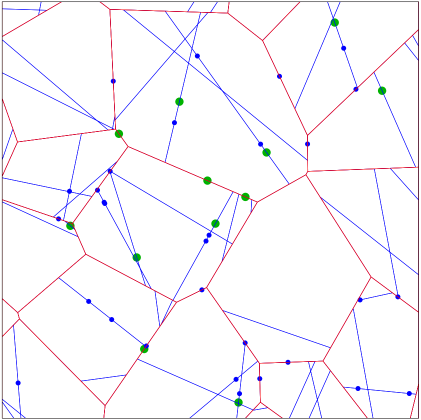

network components along roads

- main roads
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- network components:
  - higher-level components (green)
  - lower-level components (blue)

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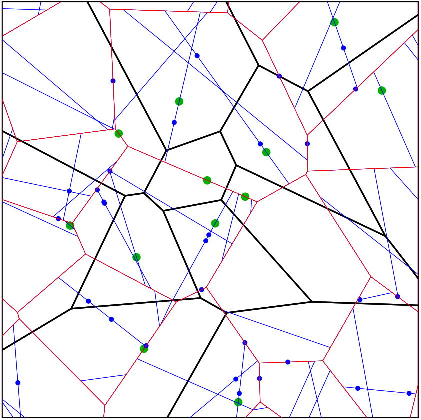

serving zones

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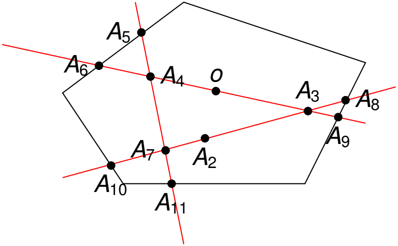
Shortest-path trees

- consider segment system S^*_H of a typical serving zone



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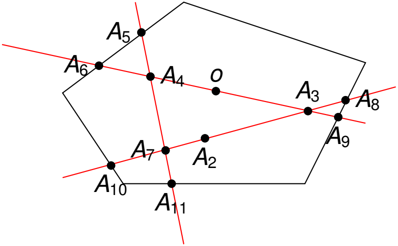

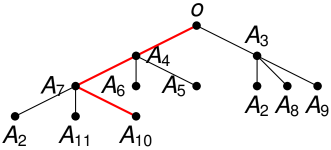

Shortest-path trees

- consider segment system S_H^* of a typical serving zone
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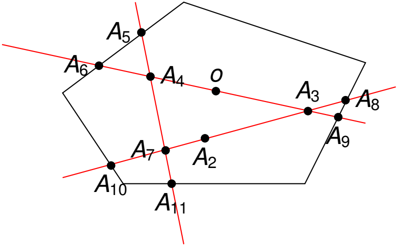
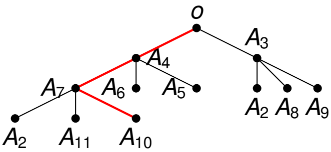
- goal. asymptotic characteristics of *backbone lengths* in unboundedly dense networks?
  - asymptotic marginal distribution of longest-branch length?
  - asymptotic joint distribution of longest-branch lengths in both subtrees?





Applications: boundedness of cells & shape theorem

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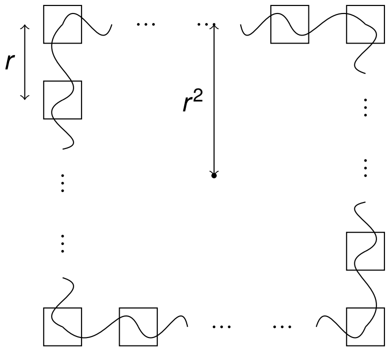
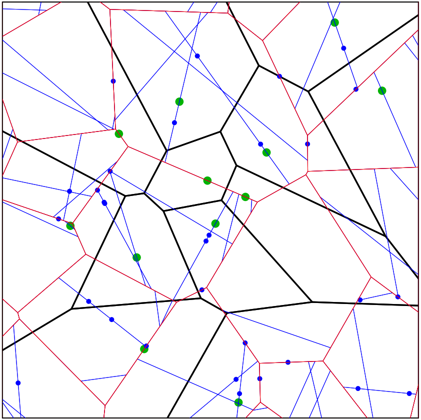
- idea: use theorem to construct closed curve of edges around the origin
- consider annulus of squares of side length $\sqrt{r}$ at distance r^2 from o and distance r from each other
- by the thm: points in neighboring squares can be connected by a path far away from o whp
- $\Rightarrow$ claim follows from Borel-Cantelli lemma



r
r^2

Stochastic Subscriber Network



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- $Z(\lambda) = \sup_{x \in S_\lambda^*} \ell(o, x)$

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- distribution of R explicitly known, see Calka 2002

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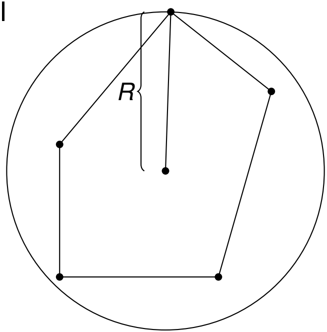

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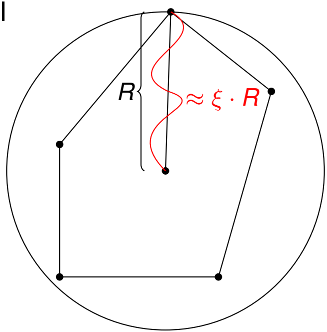

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- *then* $\lim_{k\to\infty} M_k = M < \infty$ *a.s. and* $\exists C_1, C_2 > 0$ *such that*

$$\mathbb{P}\left(|M| \geq x\sqrt{x_0}\right) \leq C_1 \exp(-C_2 x) \qquad \text{for all } x \leq x_0$$

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- based on martingale concentration inequality due to Kesten (1993)

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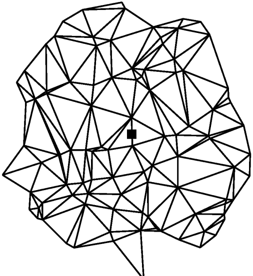

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