Zeros of Random Analytic Functions

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September 3, 2013

Algebraic equations

Consider a polynomial of degree n:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0.$$

Here, $a_0, a_1, \ldots, a_n$ are complex numbers. Assume that $a_n \neq 0$.

Fundamental theorem of algebra

The equation $P(x) = 0$ has at least one complex solution.

Corollary

There is a decomposition of the form

$$P(x) = a_n(x - x_1)(x - x_2)\ldots(x - x_n),$$

where $x_1, \ldots, x_n$ are complex numbers called the roots (or zeros) of P.

Algebraic equations

Example

Find the roots of the equation

$$x^n + x^{n-1} + \ldots + x + 1 = 0.$$

Algebraic equations: Example

Solution

1) Geometric progression:

$$x^n + x^{n-1} + \ldots + x + 1 = \frac{x^{n+1} - 1}{x - 1}.$$

2) Roots of $x^{n+1} - 1 = 0$:

$$1, \omega, \omega^2, \ldots, \omega^n,$$

where $\omega = e^{\frac{2\pi i}{n+1}}$.

3) Roots of $x^n + x^{n-1} + \ldots + x + 1 = 0$:

$$\omega, \omega^2, \ldots, \omega^n.$$

A random equation

Statement of the problem

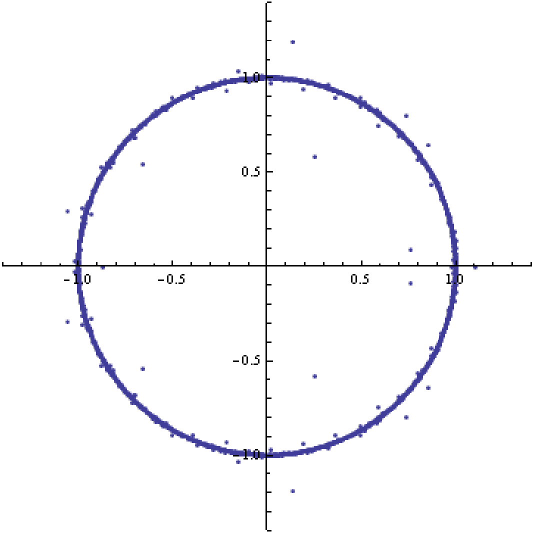
We consider an equation with **random** coefficients, for example

$$z^{2000} - z^{1999} + z^{1998} + z^{1997} - z^{1996} - \ldots + z^3 + z^2 - z + 1 = 0$$

We would like to describe the distribution of its solutions in the complex plane.

Roots of a random equation

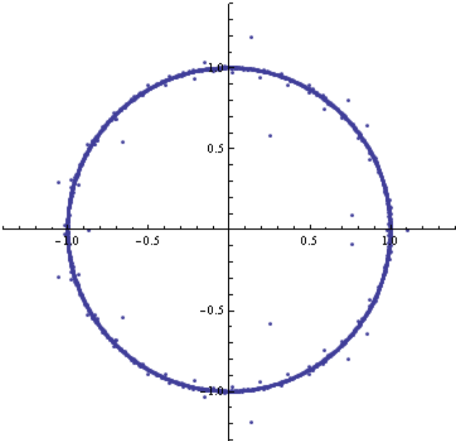
Zeros of a random polynomial of degree $n = 2000$



Roots of a random equation

Observations:

- Most roots are close to the unit circle.
- The distribution of the roots around the circle is approximately uniform.



Why are most roots close to the unit circle?

Explanation

1) Take some x with $|x| > 1$. Then,

$$|x^n| > |x^{n-1}| > |x^{n-2}| > \dots$$

Difficult to cancel...

2) Take some x with $|x| < 1$. Then,

$$1 > |x| > |x^2| > |x^3| > \dots$$

Difficult to cancel...

3) Take some x with $|x| \approx 1$

$$1 \approx |x| \approx |x^2| \approx \dots \approx |x^{n-1}| \approx |x^n|.$$

Easy to cancel...

Random polynomials (Kac ensemble)

Notation

- Let $\xi_0, \xi_1, \ldots$ be independent identically distributed random variables.
- The random polynomial of the form

$$P_n(z) := \xi_0 + \xi_1 z + \xi_2 z^2 + \ldots + \xi_n z^n$$

  is called the Kac polynomial.
- Denote by $z_1, \ldots, z_n$ the complex roots of P_n.

Empirical distribution of roots

Notation

- Let $\mathbb{M}$ be the set of probability measures on $\mathbb{C}$.
- Endow $\mathbb{M}$ with weak topology: A sequence of probability measures $\mu_1, \mu_2, \ldots \in \mathbb{M}$ converges to $\mu \in \mathbb{M}$ if for every bounded, continuous function $f : \mathbb{C} \to \mathbb{R}$ we have

$$\lim_{n\to\infty} \int_{\mathbb{C}} f d\mu_n = \int_{\mathbb{C}} f d\mu.$$

- The empirical distribution of roots of P_n is the probability measure

$$\mu_n = \frac{1}{n} \sum_{k=1}^{n} \delta(z_k).$$

- Let $\lambda \in \mathbb{M}$ be the uniform distribution on the unit circle.

Zeros of Kac polynomials

Theorem (Ibragimov und Zaporozhets, 2011)

The following conditions are equivalent:

1. μ_n converges to λ almost surely, that is
$$\mathbb{P}\left[\mu_n \xrightarrow[n\to\infty]{w} \lambda\right] = 1.$$
2. $\mathbb{E}\log(1+|\xi_0|) < \infty.$

History

Erdös and Turan (1950), Hammersley (1954), Shparo and Shur (1962), Arnold (1966), Shepp and Vanderbei (1995), Hughes and Nikeghbali (2008),...

The role of the log-moment condition

Exercise

The following conditions are equivalent:

1. $\mathbb{E}\log(1+|\xi_0|) < \infty$.
2. For **every** $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{\xi_k}{e^{\varepsilon k}} = 0 \text{ a.s.}$$

3. For **some** $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{\xi_k}{e^{\varepsilon k}} = 0 \text{ a.s.}$$

Remark

The series $\sum_{k=0}^{\infty} \xi_k z^k$ converges a.s. in the unit disk if and only if $\mathbb{E}\log(1+|\xi_0|) < \infty$.

Weyl polynomials

Definition

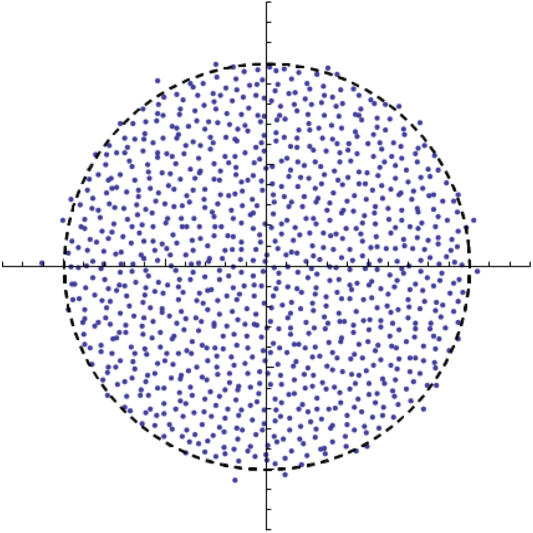
- Let $\xi_0, \xi_1, \ldots$ be independent identically distributed random variables.
- Consider the Weyl polynomials

$$P_n(z) = \sum_{k=0}^{n} \xi_k \frac{z^k}{\sqrt{k!}}.$$

- Let $z_1, \ldots, z_n$ be the zeros of P_n.

Weyl polynomials

Zeros of a Weyl polynomial



Zeros of Weyl polynomials

Theorem (Kabluchko, Zaporozhets, 2012)

The following conditions are equivalent:

1. The probability measure $\frac{1}{n}\sum_{k=1}\delta(\frac{z_k}{\sqrt{n}})$ converges a.s. to the uniform distribution on the unit circle $\{|z| \leq 1\}$.
2. $\mathbb{E}\log(1+|\xi_0|) < \infty$.

Random matrices

Statement of the problem

- Let ξ_{ij}, $i, j \in \mathbb{N}$, be independent identically distributed random variables with $\mathbb{E}\xi_{ij} = 0$, $\mathbb{E}\xi_{ij}^2 = 1$.
- Consider the matrix

$$M_n = \begin{pmatrix} \xi_{11} & \dots & \xi_{1n} \\ \vdots & \vdots & \vdots \\ \xi_{n1} & \dots & \xi_{nn} \end{pmatrix}.$$

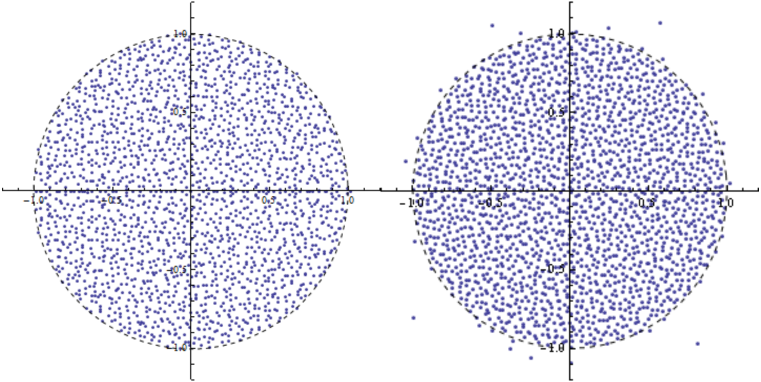
- Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of M_n.

Theorem (T. Tao und V. Vu, 2010)

The probability measure $\frac{1}{n} \sum_{k=1}^{n} \delta(\frac{\lambda_k}{\sqrt{n}})$ converges a.s. to the uniform distribution on the unit disk $\{|z| \le 1\}$.

Random matrices

Left: Eigenvalues of a random matrix
Right: Zeros of a Weyl polynomial



Littlewood–Offord polynomials (1939)

- Let $\xi_0, \xi_1, \dots$ be independent identically distributed random variables with $\mathbb{E}\log(1+|\xi_0|) < \infty$.
- Consider the Littlewood–Offord polynomials

$$P_n(z) = \sum_{k=0}^{n} \xi_k \frac{z^k}{(k!)^\alpha}.$$

- Let $z_1, \dots, z_n$ be the zeros of P_n.

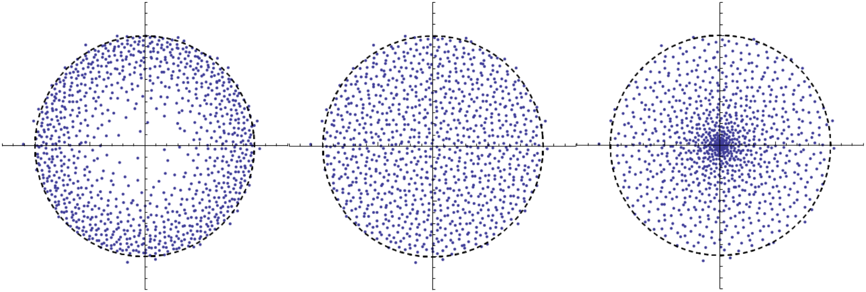
Theorem (Kabluchko, Zaporozhets, 2012)

The probability measure $\frac{1}{n}\sum_{k=1}^{n}\delta(\frac{z_k}{n^\alpha})$ converges a.s. to the probability measure with the density

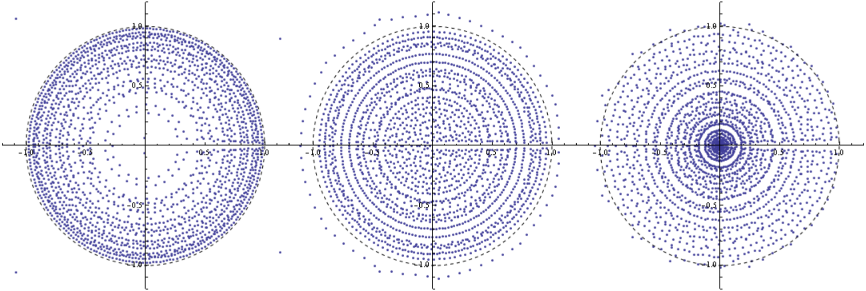
$$\frac{1}{2\pi\alpha}|z|^{\frac{1}{\alpha}-2}, \quad |z| < 1.$$

Littlewood–Offord polynomials

Zeros of the Littlewood–Offord polynomials:
Normally distributed coefficients



Zeros of the Littlewood–Offord polynomials:
Log–Pareto coefficients



Szegö polynomials

Compare: Szegö polynomials $s_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}$.

Remark

Taylor series for e^z:

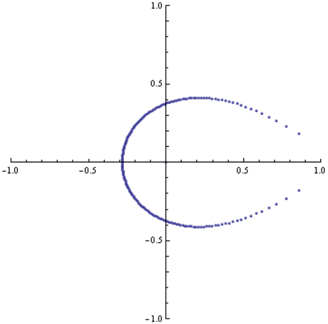
$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

Complex zeros of e^z?

Szegö polynomials

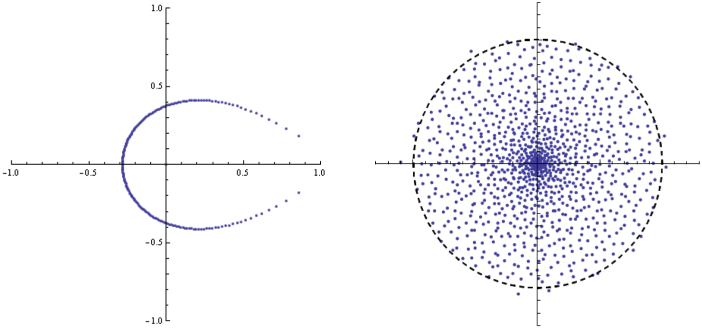
Theorem (Szegö, 1924)

The zeros of $s_n(nz)$ cluster along the curve $|ze^{1-z}| = 1$, $|z| < 1$.



Szegö and Littlewood–Offord polynomials

Zeros of $\sum_{k=0}^{n} \frac{z^k}{k!}$ and $\sum_{k=0}^{n} \xi_k \frac{z^k}{k!}$



General statement

Assumptions

Consider random polynomials (Taylor series) of the form

$$P_n(z) = \sum_{k=0}^{\infty} f_{k,n}\xi_k z^k, \quad z \in \mathbb{C}.$$

Assume that

1. $\xi_1, \xi_2, \ldots$ are independent identically distributed random variables with $\mathbb{E}\log(1 + |\xi_0|) < \infty$.
2. $f_{k,n}$ are numbers such that for every $t > 0$,

$$|f_{tn,n}| \approx e^{-nu(t)}, \quad n \to \infty,$$

where $u(t)$ is a function of $t > 0$.

Examples

Kac polynomials

Kac polynomials: $P_n(z) = \sum_{k=0}^{n} \xi_k z^k$.

- $f_{k,n} = 1$ for $k \le n$, hence $u(t) = 0$ for $t \le 1$.
- $f_{k,n} = 0$ for $k > n$, hence $u(t) = +\infty$ for $t > 1$.

Weyl polynomial

Weyl polynomials: $P_n(\sqrt{n}z) = \sum_{k=0}^{n} \xi_k z^k \sqrt{\frac{n^k}{k!}}$. For $t < 1$:

$$f_{tn,n} \approx \sqrt{\frac{n^{tn}}{(tn/e)^{tn}}} = e^{-\frac{n}{2}(t \log t - t)}.$$

Hence, $u(t) = \frac{1}{2}(t \log t - t)$ for $t < 1$ and $u(t) = +\infty$ for $t > 1$.

Laplace operator

Definition

Let $f(x, y)$ be a function of two variables. Laplace operator:

$$\Delta f(x, y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

Example

$$\Delta \log |z| = ???.$$

Poincaré–Lelong formula

Remark

We have

$$\frac{1}{2\pi}\Delta \log |z - w| = \delta(w).$$

Hence,

$$\frac{1}{2\pi}\Delta \log |(z - w_1)\dots(z - w_n)| = \delta(w_1) + \dots + \delta(w_n).$$

Theorem (Poincaré–Lelong)

Let $f(z)$ an analytic function. Then,

$$\frac{1}{2\pi}\Delta \log |f(z)|$$

is the measure counting the zeros of f.
Exception: f is identically 0.

Limiting distribution of zeros

Recall that we consider random Taylor series

$$P_n(z) = \sum_{k=0}^{\infty} f_{k,n}\xi_k z^k, \quad z \in \mathbb{C}.$$

Empirical distribution of zeros:

$$\lim_{n\to\infty} \frac{1}{2\pi n}\Delta \log |P_n(z)|.$$

Interchange lim and Δ:

$$\frac{1}{2\pi}\Delta \lim_{n\to\infty} \frac{1}{n} \log |P_n(z)|.$$

Limiting distribution of zeros

How large is

$$P_n(z) = \sum_{k=0}^{\infty} f_{k,n}\xi_k z^k?$$

Let $k = tn$, where $t > 0$:

$$|f_{k,n}\xi_k z^k| \approx e^{-nu(t)} e^{tn \log |z|} = e^{n(t \log |z| - u(t))}.$$

It follows that

$$P_n(z) \approx e^{n \sup_{t>0}(t \log |z| - u(t))} = e^{nu^*(\log |z|)}.$$

Hence,

$$\lim_{n\to\infty} \frac{1}{n} \log |P_n(z)| = u^*(\log |z|).$$

Limiting distribution of zeros

"Theorem"

The limiting empirical distribution of zeros of the random polynomial $P_n(z) = \sum_{k=0}^{\infty} f_{k,n}\xi_k z^k$ is given by

$$\frac{1}{2\pi}\Delta u^*(\log|z|).$$

Example: Kac polynomials

We have $u(t) = 0$ for $t < 1$ and $u(t) = +\infty$ for $t < 1$. Then,

$$u^*(s) = \sup_{t>0}(st - u(t)) = \begin{cases} s, & \text{if } s \geq 0, \\ 0, & \text{if } s \leq 0. \end{cases}$$

$$u^*(\log|z|) = \begin{cases} \log|z|, & \text{if } |z| \geq 1, \\ 0, & \text{if } |z| \leq 1. \end{cases}$$

The Random Energy Model

"Random Energy Model" (Derrida, 1981)

- A system can be in e^n states.
- The energy of the system in state k is $\sqrt{n}\xi_k$.
- $\xi_1, \xi_2, \ldots \sim N(0,1)$ are i.i.d. standard Gaussian random variables.
- The partition function is

$$\mathcal{Z}_n(\beta) = \sum_{k=1}^{e^n} e^{\beta\sqrt{n}\xi_k}, \quad \beta > 0.$$

The Random Energy Model

Phase transition

Free energy is given by

$$\lim_{n\to\infty} \frac{1}{n} \log |\mathcal{Z}_n(\beta)| = \begin{cases} 1 + \frac{1}{2}\beta^2, & 0 < \beta < \sqrt{2}, \\ \sqrt{2}\beta, & \beta > \sqrt{2}. \end{cases}$$

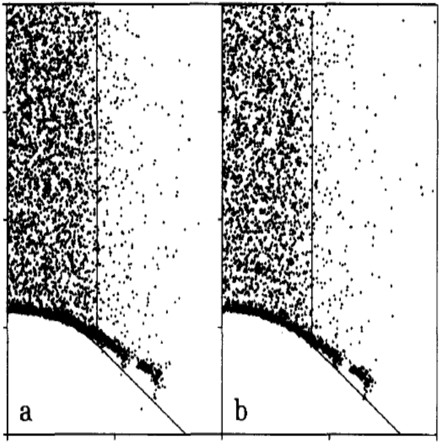
There is a phase transition at $\beta = \sqrt{2}$. Very strange because $\frac{1}{n} \log |\mathcal{Z}_n(\beta)|$ is **real analytic** for every finite system size n. Reason?

Answer (Lee and Yang)

The function $\mathcal{Z}_n(\beta)$ has **complex** zeros.

Zeros in the Random Energy Model

Complex zeros of $\mathcal{Z}_n$. Source: C. Moukarzel and N. Parga: Physica A 177 (1991).



Random Energy Model at complex temperature

Free energy for complex β

Recall that

$$\mathcal{Z}_n(\beta) = \sum_{k=1}^{e^n} e^{\beta\sqrt{n}\xi_k}, \quad \beta > 0.$$

We compute

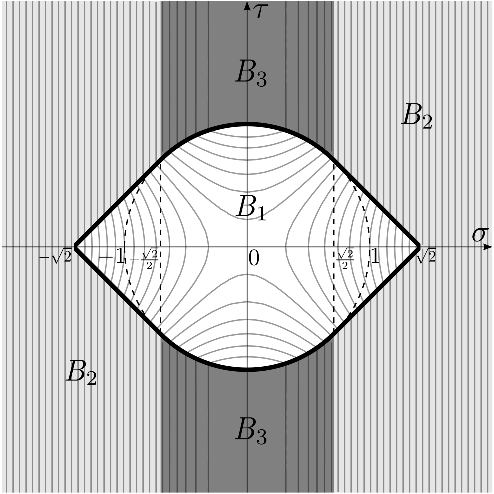
$$\lim_{n\to\infty} \frac{1}{n} \log |\mathcal{Z}_n(\beta)|.$$

Three guesses:

- $\mathcal{Z}_n(\beta) \approx \mathbb{E}\mathcal{Z}_n(\beta)$.
- $\mathcal{Z}_n(\beta) \approx \sqrt{\mathrm{Var}\mathcal{Z}_n(\beta)}$.
- $\mathcal{Z}_n(\beta) \approx \max |e^{\beta\sqrt{n}\xi_k}|$.

Phases in the Random Energy Model

The three phases

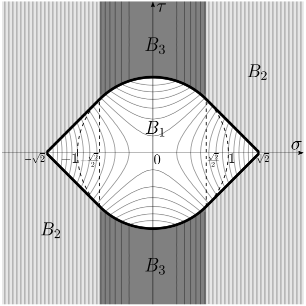


Zeros in the Random Energy Model

Theorem (Derrida, 1991; Rigorous proof: Kabluchko und Klimovsky, 2012)

For $\beta = \sigma + i\tau \in \mathbb{C}$ it holds that

$$\lim_{n\to\infty} \frac{1}{n} \log |\mathcal{Z}_n(\beta)| = \begin{cases} 1 + \frac{1}{2}(\sigma^2 - \tau^2), & \beta \in \overline{B}_1, \\ \sqrt{2}|\sigma|, & \beta \in \overline{B}_2, \\ \frac{1}{2} + \sigma^2, & \beta \in \overline{B}_3. \end{cases}$$



Poisson Zeta Function

Definition

Let $P_1, P_2, \ldots$ a Poisson Process with intensity 1. Define

$$\zeta_P(z) = \sum_{n=1}^{\infty} \frac{1}{P_n^z}.$$

The series converges for $Re\, z > 1$ a.s. since

$$\lim_{n\to\infty} \frac{P_n}{n} = 1.$$

Theorem (Kabluchko and Klimovsky, 2012)

With probability 1 this function admits a meromorphic continuation to the half-plane $Re\, z > \frac{1}{2}$.

Vielen Dank für Ihre Aufmerksamkeit!
Many thanks for your attention!