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Asymptotic properties of the parallel volume

(Based on a joint work with Markus Kiderlen)

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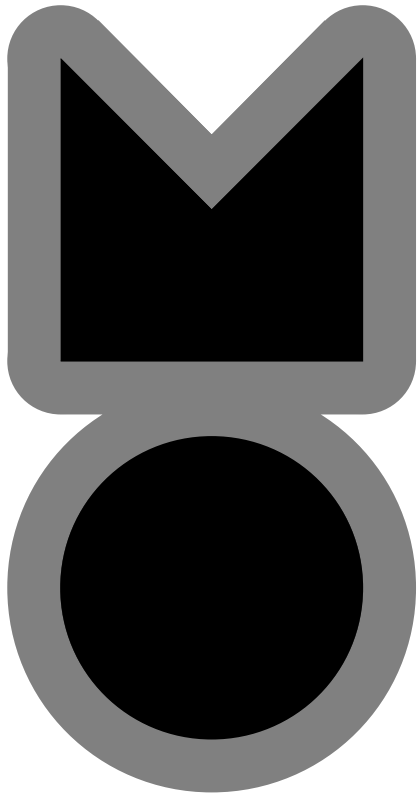
body $=$ compact, non-empty subset of $\mathbb{R}^d$

Parallel body of a body $K \subseteq \mathbb{R}^d$ at distance $r \geq 0$:

$$\{x \in \mathbb{R}^d \mid d(K,x) \leq r\} = K + rB^d,$$

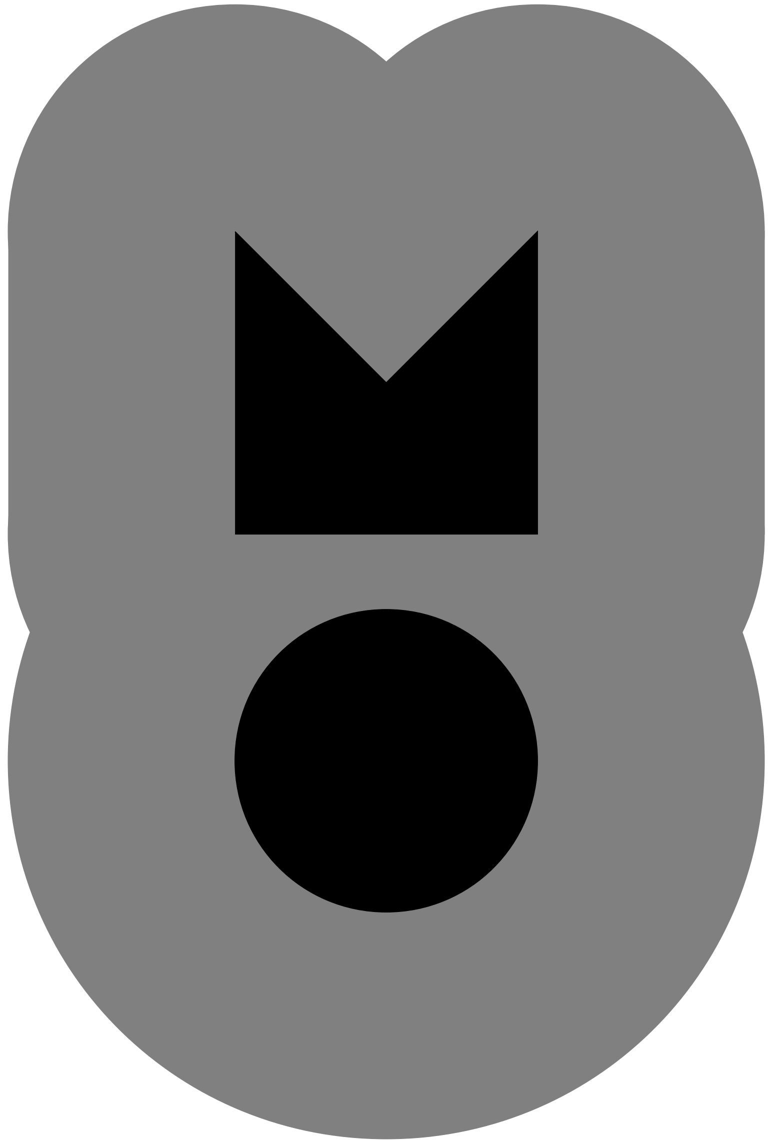
where

$$\begin{aligned} d(K,x) &:= \min\{\|y-x\| : y \in K\} \\ K+L &:= \{x+y : x \in K, y \in L\} \\ rK &:= \{rx : x \in K\} \\ B^d &:= \{x \in \mathbb{R}^d : \|x\| \leq 1\} \end{aligned}$$



K

$(K + rB^d) \setminus K$


K
$(K + rB^d) \setminus K$

Parallel volume of a body $K \subseteq \mathbb{R}^d$ at distance $r \geq 0$:

$$\lambda_d(K + rB^d)$$

λ_d = Lebesgue measure

- used to define essential concepts of convex geometry

- applications in stochastic geometry, stereology, statistics and geometric functional analysis

- well understood for convex bodies

Theorem 1 (Kiderlen & Rataj (2006))
Let $K \subseteq \mathbb{R}^d$ be a body. Then for $r \to \infty$

$$\lambda_d(\operatorname{conv} K + rB^d) - \lambda_d(K + rB^d) \in o(r^{d-1})$$

Theorem 2 (K. (2009))

$$\lambda_d(\operatorname{conv} K + rB^d) - \lambda_d(K + rB^d) \in O(r^{d-3})$$

Theorem 3

Let $K \subseteq \mathbb{R}^d$ be a body. Then

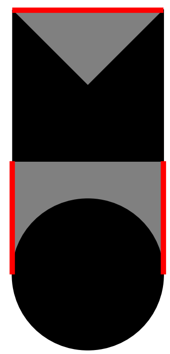
$$\lim_{r\to\infty} \frac{\lambda_d(\operatorname{conv} K + rB^d) - \lambda_d(K + rB^d)}{r^{d-3}} = \frac{(d-1)\kappa_{d-1}}{2} V_1^{(3)}(K),$$

where $\kappa_j := \lambda_j(B^j)$.

What is the intrinsic power volume $V_1^{(3)}(K)$?

For $d = 2$:

Consider $(\operatorname{bd} \operatorname{conv} K) \setminus K$.



K

$(\mathrm{conv}\, K) \setminus K$

$(\mathrm{bd}\, \mathrm{conv}\, K) \setminus K$

$(\mathrm{bd}\, \mathrm{conv}\, K) \setminus K$ consists of countably many disjoint, relatively open segments

Call this collection $\mathcal{F}^*(K)$

Theorem 3 (Recalled)
Let $K \subseteq \mathbb{R}^d$ be a body. Then

$$\lim_{r\to\infty} \frac{\lambda_d(\operatorname{conv} K + rB^d) - \lambda_d(K + rB^d)}{r^{d-3}} = \frac{(d-1)\kappa_{d-1}}{2} V_1^{(3)}(K),$$

where $\kappa_j := \lambda_j(B^j)$.

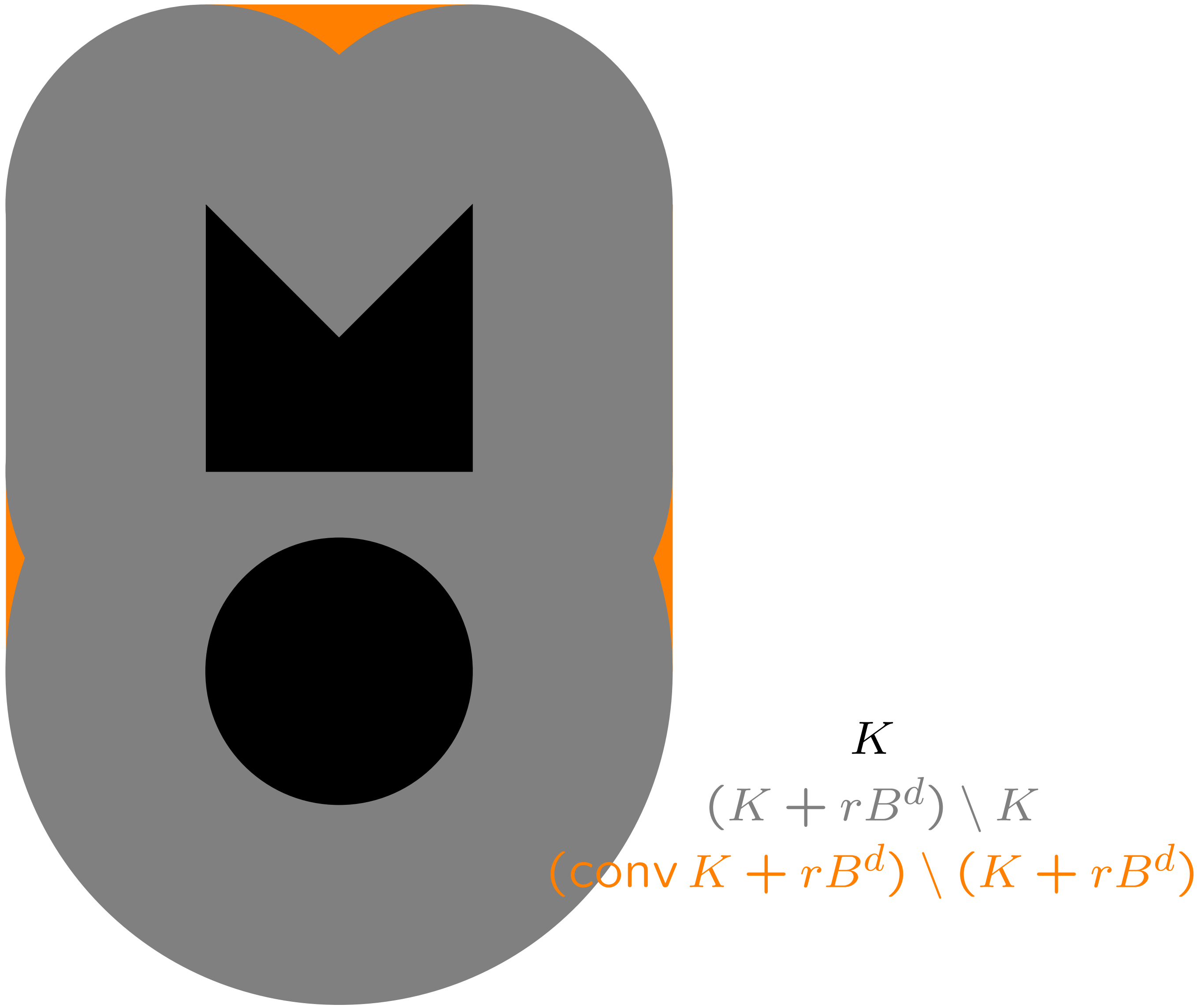
For $d = 2$:

$$V_1^{(3)}(K) = \frac{1}{24} \sum_{F \in \mathcal{F}^*(K)} (\ell(F))^3,$$

where $\ell(F)$ is the length of F.

For $d \geq 2$:

$$V_1^{(3)}(K) = \frac{1}{\kappa_{d-1}} \int_{\operatorname{bd}\operatorname{conv} K} \| \dots \|^2 \, C_1(\operatorname{conv} K, dx)$$


K
$(K + rB^d) \setminus K$
$(\operatorname{conv} K + rB^d) \setminus (K + rB^d)$

Proof: $d = 2$

$$\lambda_d(\mathrm{conv}\, K + rB^d) - \lambda_d(K + rB^d)$$
$$= \sum_{F \in \mathcal{F}^*(K)} \lambda_d(\{x \in (\mathrm{conv}\, K + rB^d) \setminus (K + rB^d) \mid p(K, x) \in F\}),$$

where $p(K, x)$ = metric projection of x onto K
= point in K closest to x

For $r \geq \ell(F)/2$

$$\begin{aligned}\lambda_d(\{x \in (\text{conv}\, K + rB^d) \setminus (K + rB^d) \mid p(K,x) \in F\}) \\ \leq 2\int_0^{\ell(F)/2} r - \sqrt{r^2 - x^2}\, dx \\ = \tfrac{1}{24}\ell(F)^3 r^{-1} + O(r^{-2})\end{aligned}$$

$\forall_{\delta \in (0,\ell(F))} : \exists_{r_0 > 0} : \forall_{r > r_0} :$

$$\begin{aligned}\lambda_d(\{x \in (\text{conv}\, K + rB^d) \setminus (K + rB^d) \mid p(K,x) \in F\}) \\ \geq 2\int_0^{(\ell(F)-\delta)/2} r - \sqrt{r^2 - x^2}\, dx \\ = \tfrac{1}{24}(\ell(F) - \delta)^3 r^{-1} + O(r^{-2})\end{aligned}$$

Theorem 3 (Recalled)
Let $K \subseteq \mathbb{R}^d$ be a body. Then

$$\lambda_d(\operatorname{conv} K + rB^d) - \lambda_d(K + rB^d) = \frac{(d-1)\kappa_{d-1}}{2} V_1^{(3)}(K) r^{d-3} \pm o(r^{d-3}),$$

where $\kappa_j := \lambda_j(B^j)$.

Can we go further?

In general: no

For finite sets $K \subseteq \mathbb{R}^d$ fulfilling a certain geometric condition:

$$\lambda_d(\text{conv}\, K + rB^d) - \lambda_d(K + rB^d) = \sum_{j=3-d}^{\infty} a_j(K) r^{-j}$$

for sufficiently large r, where

$$a_j(K) = \sum_{\substack{i=1, \\ 2|(j+d-i)}}^{\substack{\min\{d-1, \\ j+d-2\}}} (-1)^{(j+d-i)/2} \binom{(d-i)/2}{(j+d-i)/2} \kappa_{d-i}$$

$$\sum_{F \in \mathcal{F}_i(\text{conv}\, K)} \gamma(F, \text{conv}\, K) \int_F d(K \cap F, x)^{j+d-i}\, dx.$$

Literature:

1. Kampf, J.; Kiderlen, M.: *Large parallel volumes of finite and compact sets in d-dimensional Euclidean space*, Documenta Mathematica **18** (2013), 275–295.