The bifurcation diagrams of some Hamiltonian systems

(differential geometry)

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Description of the investigated systems

1) Assume that we have a function $f(r)$: $f(r) > 0, r \in (a, b)$, where a, b – finite points. $f(r)$ – smooth function on (a, b).

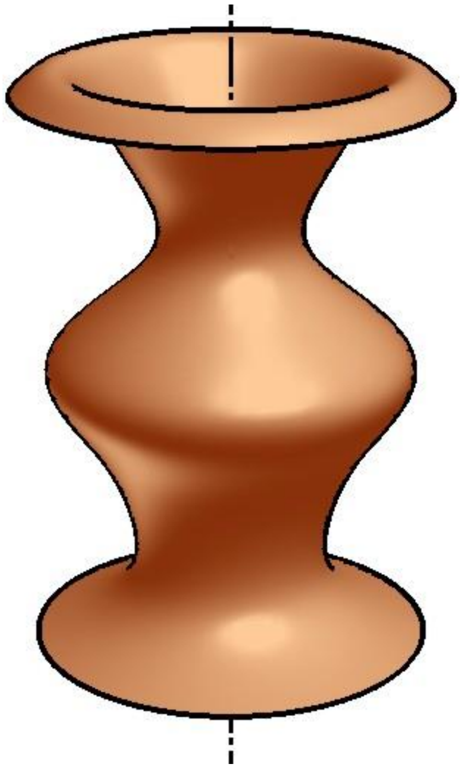
   Then we rotate this function around the axis Or.

   We obtain a surface $M^2 \approx (a, b) \times S^1$,

   which is called the ***surface of revolution***.

2) Let $V(r)$ be a smooth function on (a, b).

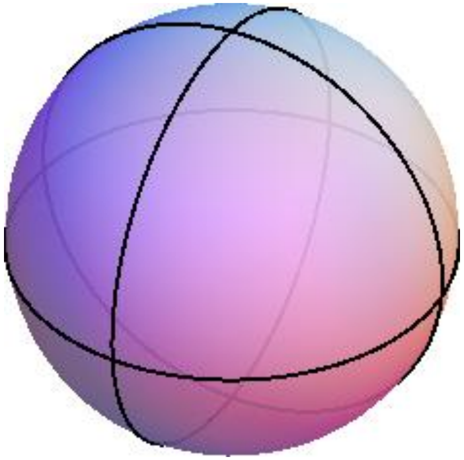
   Let us call it a ***potential function***.



<u>Our first goal is to describe the motion of a point moving on this surface under the action of a potential.</u>

Statement of the problem

Definition. Assume that we have a surface with a riemannian metric. The shortest curve on it (in the sense of metric) between two points is called a *geodesic line* (or *geodesic*).



Example. Geodesics on a sphere are the great circle arcs.

According to classical mechanics the studied motion is equivalent to the behavior of geodesic flows on a given surface under potential force.

Definition. Since we can describe the system on M^2, defined by $(f(r), V(r))$, using the Hamiltonian equations $\dot{p}_i = -\frac{\partial H}{\partial q_i}, \dot{q}_i = \frac{\partial H}{\partial p_i}, i = 1,2$, where q_i are the coordinates of the point, p_i are its impulses, t is time and $H = H(q, p, t)$ is a Hamiltonian function (energy of the system), then the investigated system is called a ***Hamiltonian system on the surface of revolution***.

Definitions (1)

Definition. Assume that we have a smooth manifold M^{2n} with a closed non-degenerated differential 2-form ω on it (*symplectic form*). Then (M^{2n}, ω) is called a ***symplectic manifold***.

Definition. Let us define a vector field $sgradH$ in local coordinates $x^1, \dots, x^{2n}$ on (M^{2n}, ω) with the equations:

$$(sgradH)^i = \omega^{ij} \frac{\partial H}{\partial x^j},$$

where ω^{ij} is the inverse matrix to the symplectic form.

Then *we can consider the studied system as a symplectic manifold*

$(M^2, \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$ *with the Hamiltonian vector field* $v = sgradH$.

Definitions (2)

<u>Definition.</u> The Hamiltonian system on the manifold $(M^{2n}, v = sgradH)$ is called ***Integrable in Liouville sense***, if there exists a set of functions $f_1, \dots, f_n$*:*

1) $f_1, \dots, f_n$ *are the first integrals of* v*;*

2) $gradf_1, \dots, gradf_n$ *are linearly independent almost everywhere on* M^{2n}*;*

3) $\{f_p, f_q\} := \omega^{ij} \frac{\partial f_p}{\partial x^i} \frac{\partial f_q}{\partial x^j} = 0$ *for any* p *and* q*;*

4) *vector fields* $sgradf_i$ *are complete.*

The phase space of a studied system *(f(r),V(r))* has a dimension of 4 with cylindrical coordinates $(\mathrm{r}, \mathrm{p_r}, \varphi, \mathrm{p_\varphi})$, and the system has 2 integrals of motion: energy $\mathbf{H}$ and $\mathbf{M} = \mathrm{p_\varphi}$ - projection of kinetic moment on axis Or.

<u>Statement.</u> This system is *Integrable in Liouville sense*.

In particular, the closures of integral trajectories form the tori (we call it the *Liouville tori*).

The main problem

__Definition.__ Two Hamiltonian systems are called ***topologically equivalent***, if there is a map which transfers the closure of integral trajectories of the first system to the closure of integral trajectories of the second system.

<u>*Our main problem is to classify all Hamiltonian systems on surfaces of revolution according to their topological equivalence.*</u>

<u>How to solve this problem?</u>

Fomenko-Zieschang Theorem states that two systems are topologically equivalent if and only if their molecules coincide.

Molecules are special invariants, whose graphs correspond to the types of the singularities in their vertices, and they have the marks on their edges. These molecules are called the *marked molecules*.

So, we want to calculate these invariants for studied systems, and I did it in some important cases.
To calculate a molecule, we need to construct a ***bifurcation diagram*** *of the system.*

Definitions (3)

The equation for Hamiltonian *H* is:

$$H = \frac{{p_r}^2}{2} + \frac{{p_\varphi}^2}{2f^2(r)} + V(r) \qquad (1)$$

Definition. *Map F: M* $\rightarrow \mathbb{R}^2$:

$(r, p_r, \varphi, p_\varphi) \rightarrow (H(r, p_r, \varphi, p_\varphi), M(r, p_r, \varphi, p_\varphi))$ is called a ***momentum map***.

Definition. If $rk\ dF(x)<2$, then x is a *critical point* of F, $F(x)$ - *critical value.*

A set of critical values $\Sigma = \{F(x) \mid dF(x) < 2\}$ is called a ***bifurcation diagram***.

Then the condition on Σ is the following: linear dependence of $grad\ H$ on $grad\ M = grad p_\varphi$.

Statement. The bifurcation diagram is given by these equations:

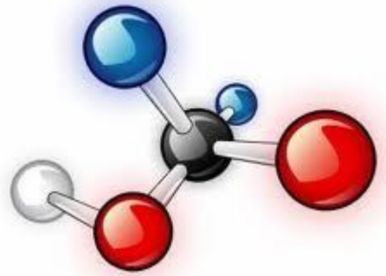
$$H = \frac{f(r)V\prime(r)}{2f\prime(r)} + V(r),$$

$$p_\varphi = \pm\sqrt{\frac{f^3(r)V'(r)}{f'(r)}} \qquad (2)$$

Remark. The bifurcation diagram is symmetric relative to the axis H.

What can we learn from the bifurcation diagram?

- According to *Liouville Theorem*, the phase space of the Integrable system is foliated into tori. This means that when we fix a level of energy H, then the submanifold of the phase space with energy H is a union of certain number of tori. There are also some critical values of H, at which submanifolds are more complicated.
- The preimage of every point (H, p_φ) in $Im\varphi$ is a torus or a union of tori.
- The bifurcation diagram foliates the image of momentum map into the cameras, and in every camera we have a constant number of tori.
- Since the curve of the bifurcation diagram corresponds to a bifurcation, then its preimage is a part of critical submanifold. We can determine the type of critical submanifold (and then – the number of tori in every camera) by examining the *effective potential* function, which we use to construct the bifurcation diagram.
- When we have constructed the bifurcation diagram and have defined a number of tori in every camera, we can construct a *molecule* – one of the main invariants of the system (a molecule consists of *vertices*, which are names of the singularities and ribs).



What can we learn from the bifurcation diagram? (2)

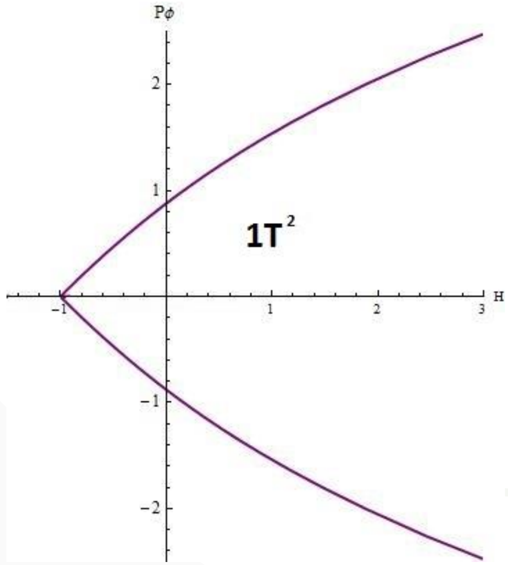
Scheme of studying the bifurcation diagram:

construction of a bifurcation diagram → examination of bifurcation curves → determination of the number of tori in each camera → construction of a molecule

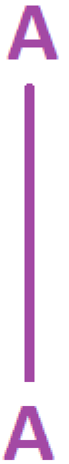
Example of a molecule (A spherical pendulum system).

Rotation function: $f(r) = \sqrt{1 - r^2}$, *potential:* $V(r) = r, r \in (-1,1)$.

bifurcation diagram:



molecule:



About the boundary of ImF

- Let us fix p_{φ}. When p_r grows, H also grows (see f.1), so there is no upper boundary of ImF.
- Conditions on the minimum of ImF:

  a) $p_r = 0 \; (\frac{\partial H}{\partial p_r} = 0)$

  b) no restrictions on φ (H doesn't depend on φ)

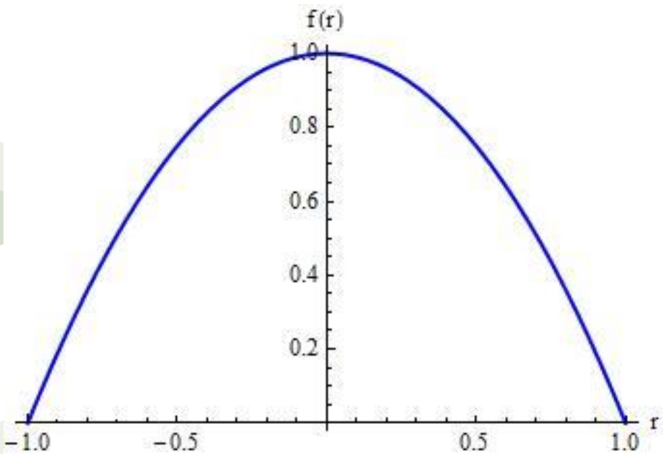
  c) $\frac{\partial H}{\partial r} = -\frac{f'(r) p_{\varphi}{}^2}{f^3(r)} + V'(r)$. We have two cases:

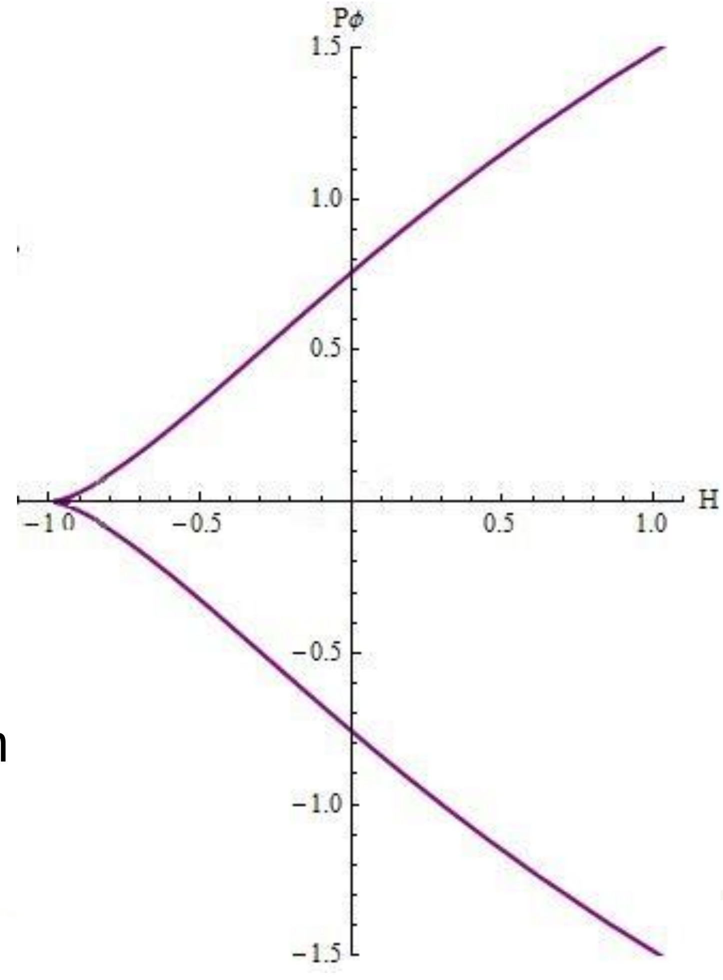
  - Exists $r_0 \in (a, b)$: $H'(r_0) = 0$ and $H(r_0)$ is a minimum. Then the lower boundary of ImF is a point which belongs to the bifurcation curve.
  - Min $H_{[a,b]} = H(a) \; or \; H(b)$ (let $r = a$). Then minimum doesn't belong to the bifurcation curve. In this case the lower boundary of ImF is a variety $\{(p_{\varphi,} H(a, \varphi, 0, p_{\varphi}))\}$. According to (1), it is a parabola (let us call it a *dotted parabola*).

<u>Conclusion.</u> Lower boundary of ImF that correspond to a fixed value of p_{φ} is a point on the bifurcation diagram or a dotted parabola.

A system with the first type of boundary of *ImF*

Rotation function: $f(r) = 1 - r^2$, *potential:* $V(r) = -r, r \in (-1,1)$.

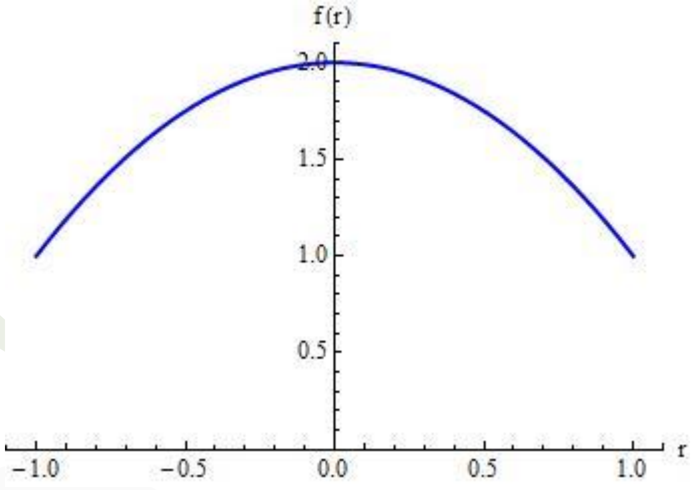


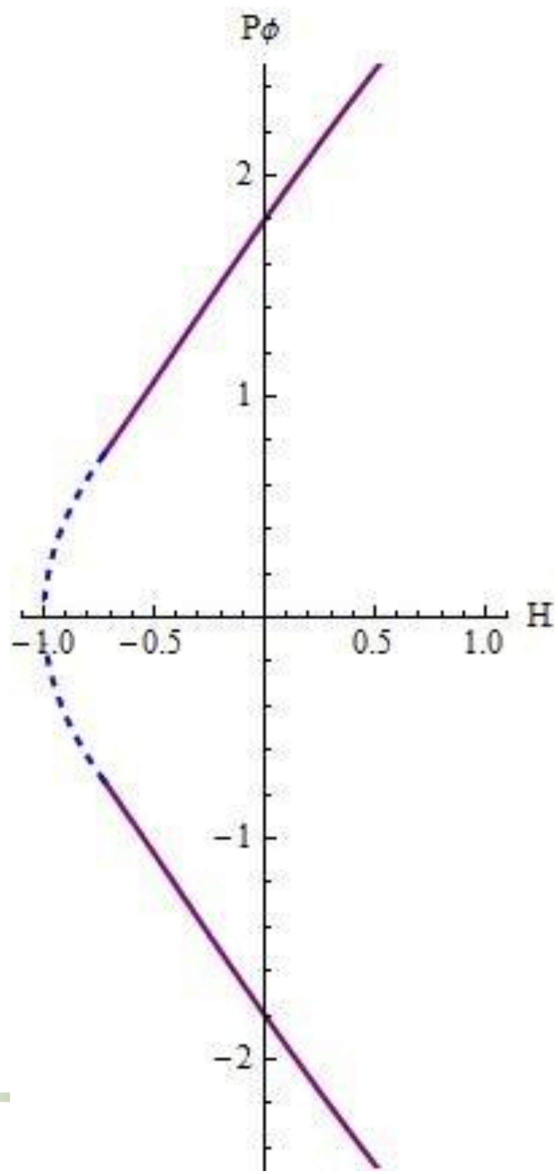


all points of lower boundary belong to the bifurcation diagram

A system with the second type of boundary of *ImF*

Rotation function: $f(r) = 2 - r^2$, *potential:* $V(r) = -r, r \in (-1,1)$.





part of the lower boundary
is a dotted parabola

About the tendency towards infinity of the bifurcation curves

Question. Let us examine the infinite curve of bifurcation diagram $(H(r), p_\varphi(r))$. Assume that we have $H(r) \to \infty$ when $r \to r_0$. What can we say about the tendency towards infinity of $p_\varphi(r)$ when $r \to r_0$?

According to formulas 2 of bifurcation diagram curves, it seems that $p_\varphi(r)$ also tends towards infinity. But this is not true. There exist two particular cases when $r \to r_0$:

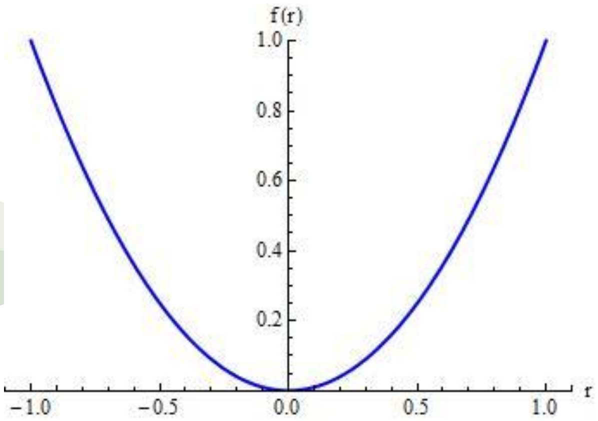
a) $H(r) \to \infty,\ p_\varphi(r) \to p_{\varphi_0}$

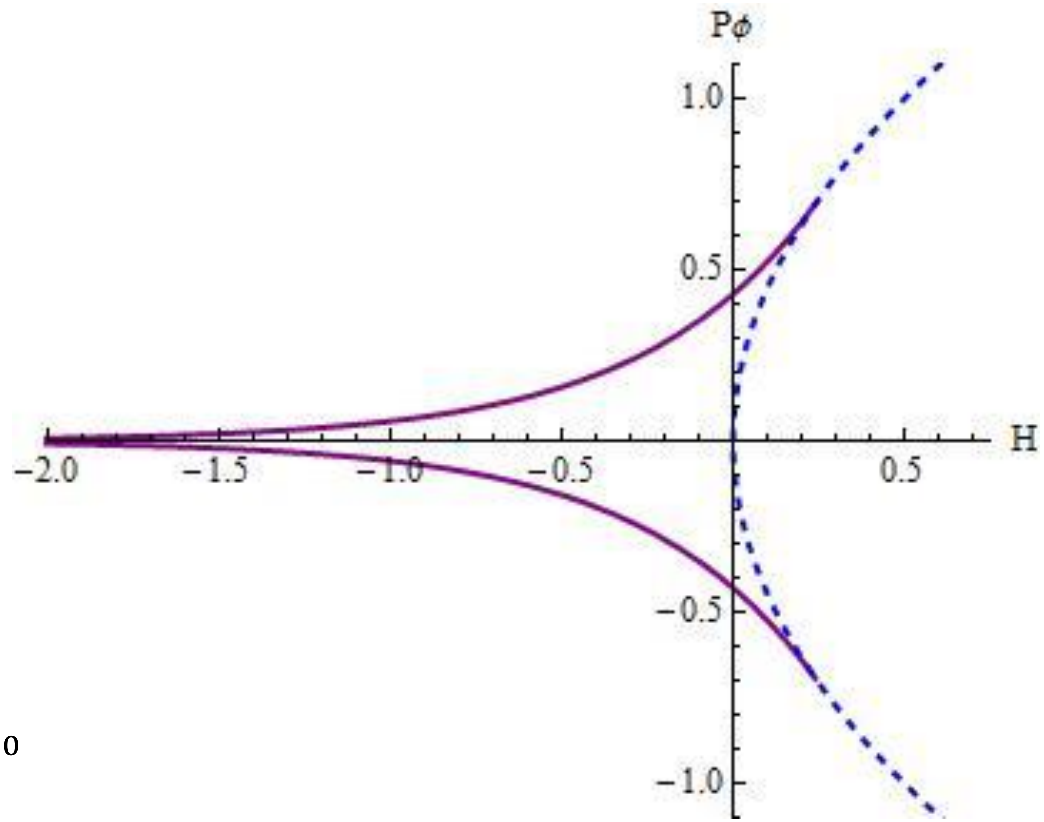
b) $H(r) \to H_0, p_\varphi(r) \to \infty$

Theorem. Let $r \to r_0$. If and only if $\{V' \to 0, f^3 V' \to \infty$ faster than $f' \to \infty\}$, or $\{V' \to V'_0 \neq 0, f \to \infty\}$, then the curve of bifurcation diagram, defined on $r_0 \in (r_1, r_2)$ tends to infinity as follows: $p_\varphi \to \infty$, $H \nrightarrow \infty$.

Example of the system of type a)

Rotation function: $f(x) = x^2$, *potential:* $V(x) = \ln x, x \in (0,1)$.



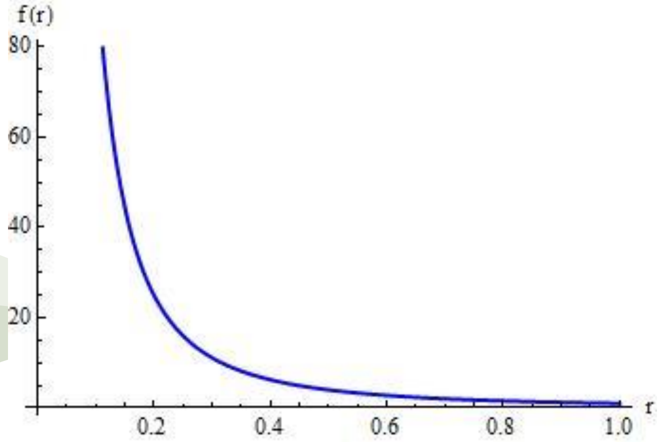


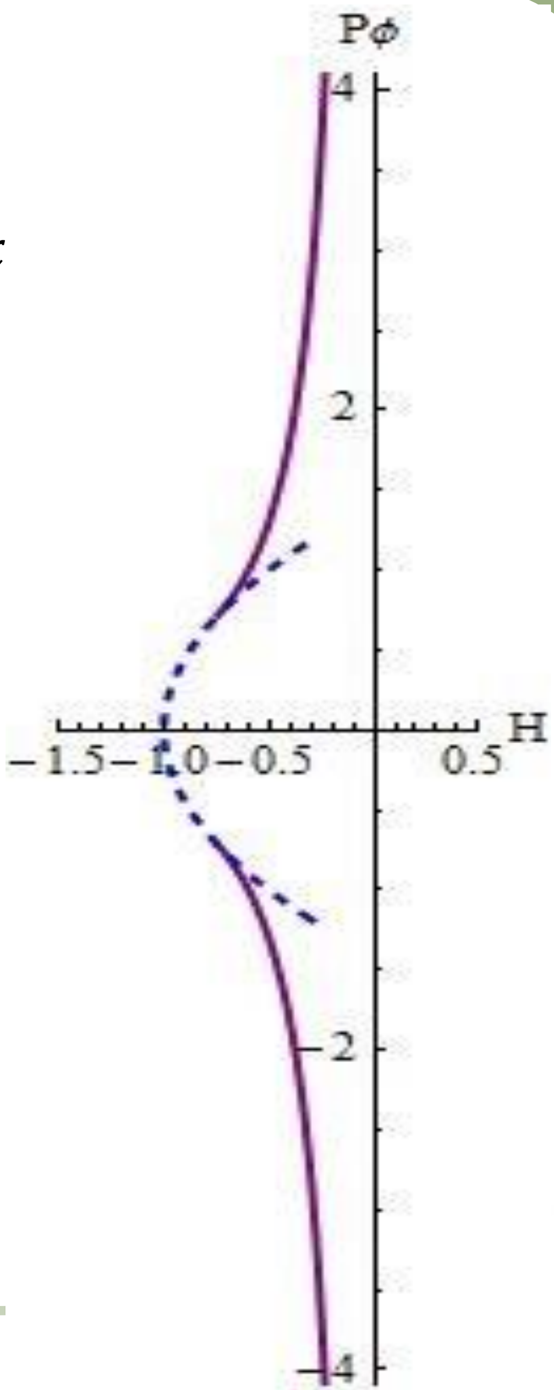
a) $H(r) \to \infty, p_\varphi(r) \to p_{\varphi_0}$

Example of the system of type b)

Rotation function: $f(x) = \frac{1}{x^2}$, *potential:* $V(x) = -x$

$x \in (0,1)$.

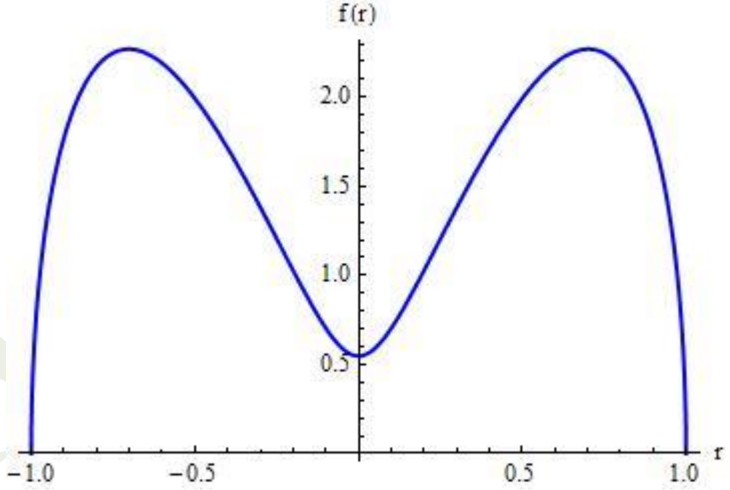


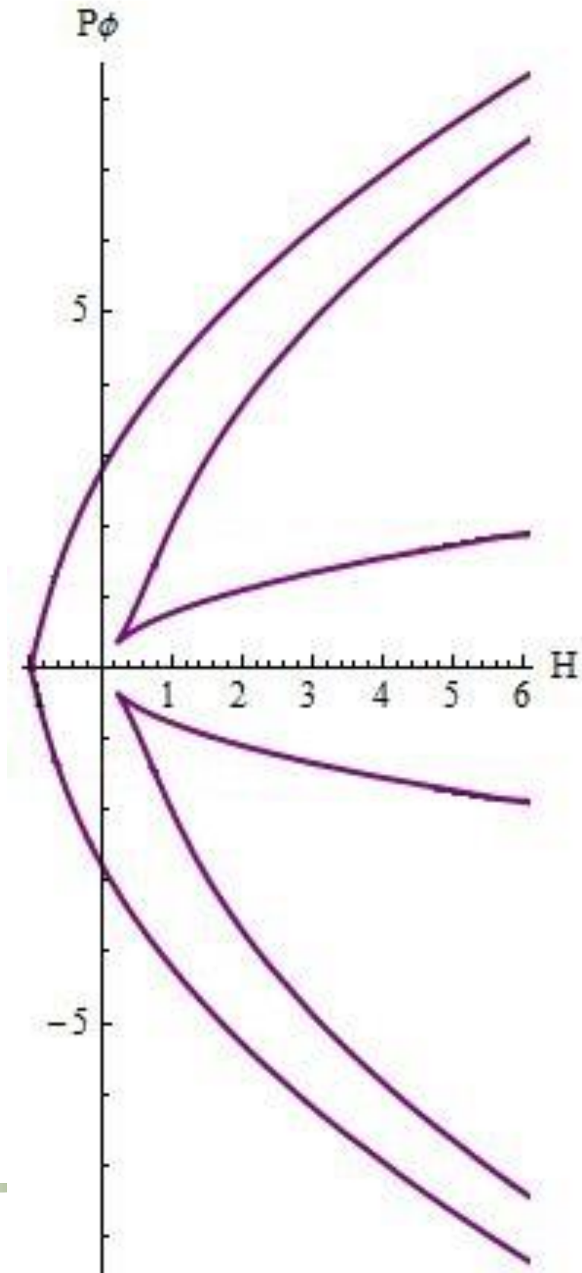


b) $H(r) \to H_0, p_\varphi(r) \to \infty$

Examples of bifurcation diagrams with “beaks”. Ex.1

Rotation function: $f(r) = \sqrt{(1 - r^2)(0.3 + 20r^2)}$,
potential: $V(r) = r, r \in (-1,1)$.

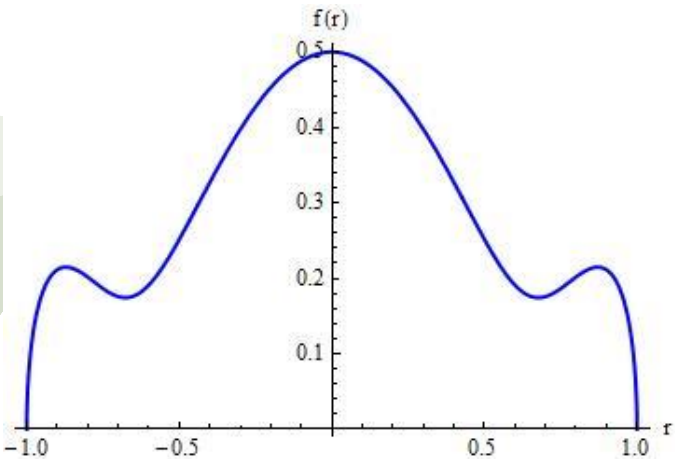


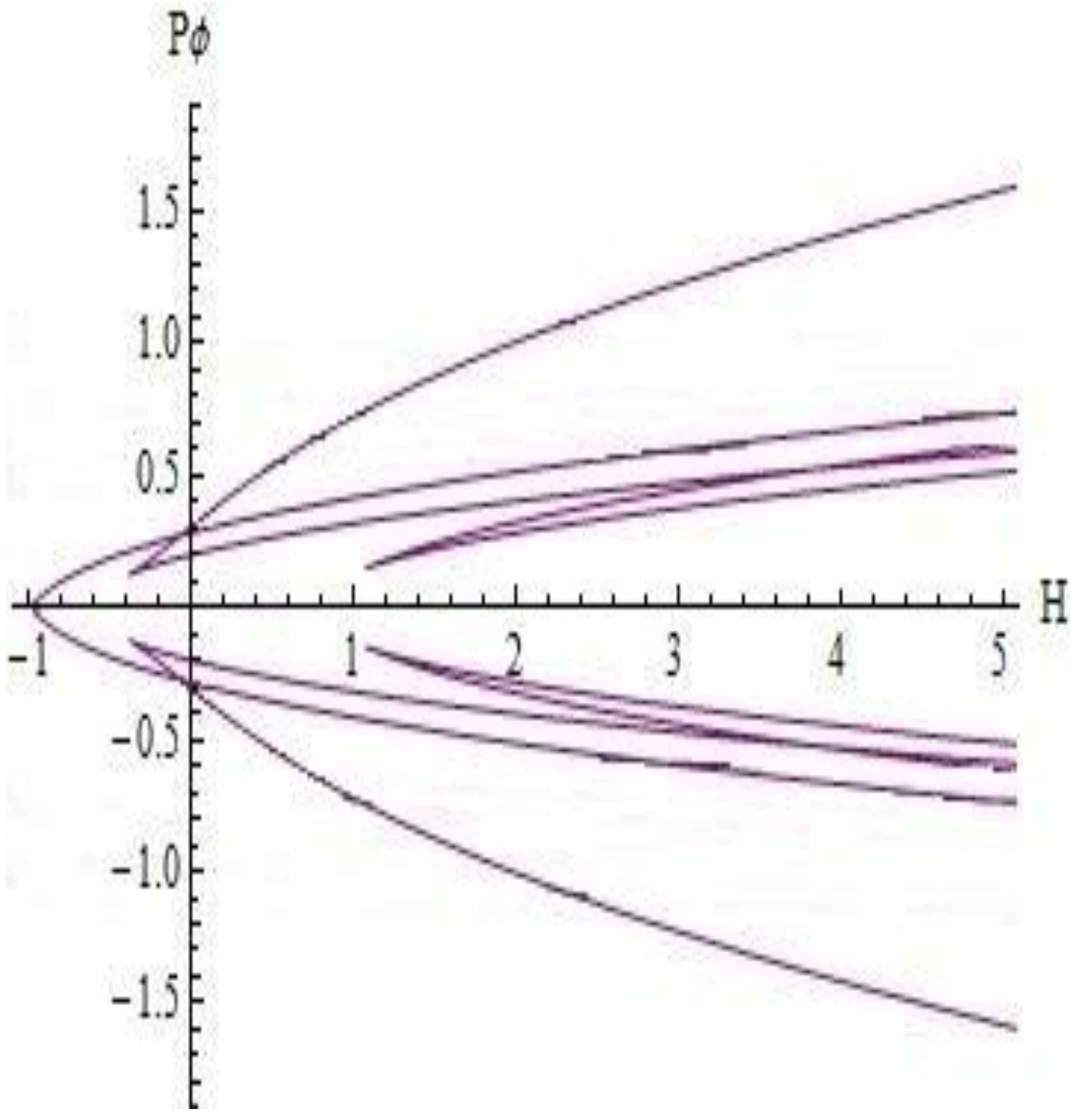


2 local maximums of $f(r)$ and 2 beaks
on bifurcation diagram

Example 2

Rotation function: $f(r) = \sqrt{(1-r^2)(1.15r^4 - 0.95r^2 + 0.25)}$,
potential: $V(r) = r, r \in (-1,1)$.





3 local maximums of *f(r)* and
4 beaks on bifurcation diagram

About the curves of the bifurcation diagram

Let us fix a gravitational potential $V(r) = r$. Assume that function $f(r)$ defined on $(-1,1)$: $f(\pm 1) = 0$.

We define an effective potential:

$$Ueff(r) = \frac{p_\varphi^2}{2f^2(r)} + V(r) \qquad (3)$$

and an auxiliary function *g(r)*:

$$g(r) = 3f^2(r)f'(r) - f^3(r)f''(r).$$

Let us denote as $(-1, r_1)$ and $\left(r_{1_i}, r_{2_i}\right), i = 1, \dots, k$, a set of intervals, at which the bifurcation diagram is defined.

Theorem. If the equation $g(r) = 0$ has no roots on $(-1, r_1)$ and has only one root on every interval $\left(r_{1_i}, r_{2_i}\right)$, then the bifurcation diagram of the system consists of one "parabola" and $2k$ beaks, where k is the number of local maximums of $f(r)$, reduced by one.

Remark. Term "parabola" means a curve constructed of two parts:

$F(H) \cup -F(H)$, where $F(H)$ is a monotonic function:

$F(H_0) = 0$, $F(H) > 0$ where $H > H_0$; and $F(H) \to \infty$ when $H \to \infty$.

Conditions on the existence of the beak

Remark. The term *beak* means a semicubical point of return.

Theorem. Assume that there exists a point ($p_{\varphi 0}, r_0$), where the following conditions are true:

$$\frac{\partial Ueff(p\varphi_*, r^*)}{\partial r} = 0,$$

$$\frac{\partial^2 Ueff(p\varphi_*, r^*)}{\partial r^2} = 0, \qquad (4)$$

$$\frac{\partial^3 Ueff(p\varphi_*, r^*)}{\partial r^3} \neq 0,\ \varepsilon_1 := sgn\left(\frac{\partial^3 Ueff(p\varphi_*, r^*)}{\partial r^3}\right),$$

$$\frac{\partial^2 Ueff(p\varphi_*, r^*)}{\partial r \partial \varphi} \neq 0,\ \varepsilon_2 = \varepsilon := sgn\left(\frac{\partial^2 Ueff(p\varphi_*, r^*)}{\partial r \partial \varphi}\right),$$

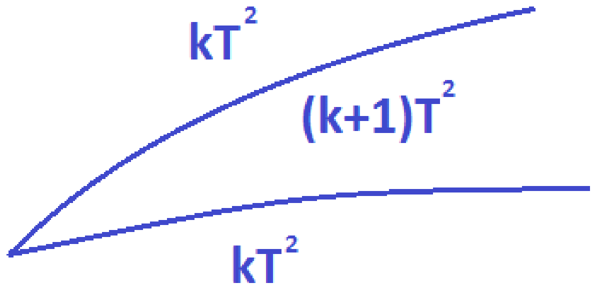
then:

a) In the neighbourhood of the point $(p\varphi_*, r^*)$ exist such local coordinates $(\widetilde{I}, \widetilde{p\varphi})$, that the curve of bifurcation diagram has a form

$$\gamma = \left\{\left(\pm\frac{2}{3\sqrt{3}}(-\varepsilon\tilde{I})^{\frac{3}{2}} + f_0(\tilde{I}), \tilde{I}\right) \middle| \varepsilon\tilde{I} < 0\right\}.$$

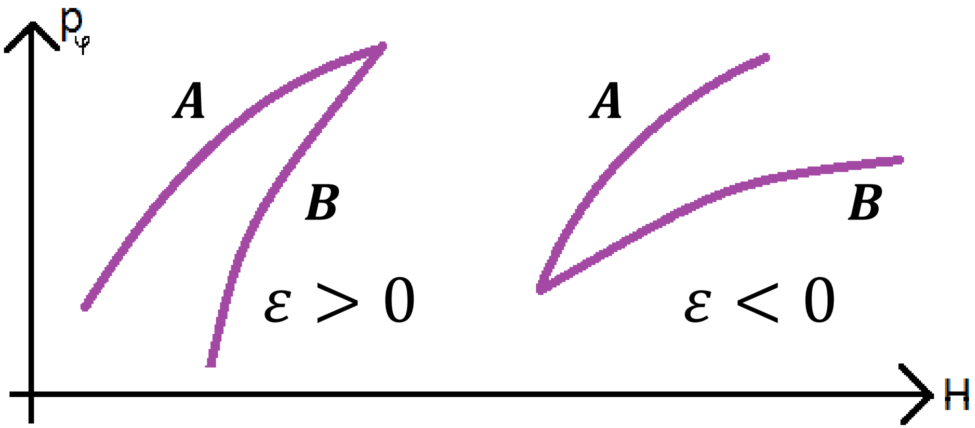
Conditions on the beak (2)

By analyzing the graph of bifurcation diagram γ, we can conclude that in the interior of the beak the number of tori increase by 1.



b) According to the $sgn(\varepsilon)$ we have two types of beaks:

Lower boundary of each beak has a singularity of type B (atom B) in its preimage. Upper boundary has a singularity of type A.

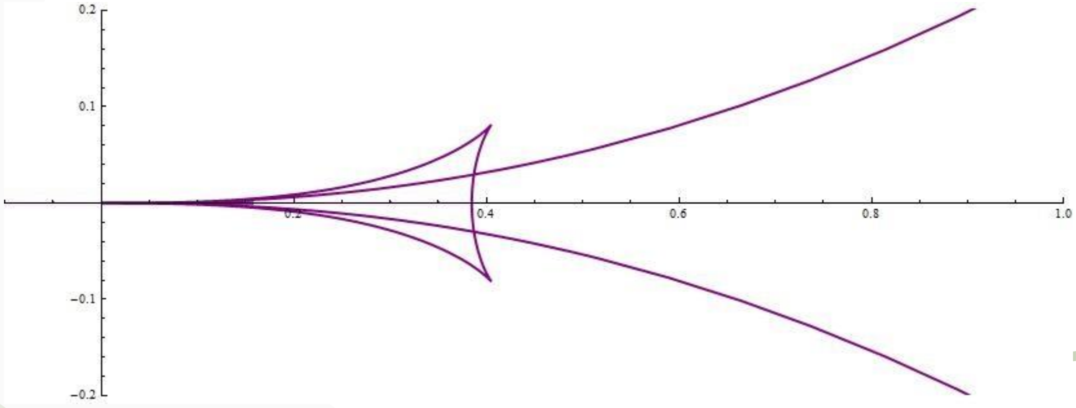


Results

- For each Hamiltonian system with $V(r) = r$ and $f(r)$: $f(r) > 0\ \forall r \in (a, b)\ and\ f(a) = f(b) = 0$, we can construct a bifurcation diagram and, hence, its molecule, using the theorems from this presentation.
- For each Hamiltonian system with another potential $V(r)$and another function $f(r)$, we can also construct a bifurcation diagram, but it can be more difficult, because the domain of definition of this diagram in general case depends on $V'(r)$; and in the case when $Ueff(p_\varphi, r)$ has more than one minimum on each interval of definition, we obtain additional bifurcation curves.

Example.

Rotation function: $f(r) = r^2$, *potential:* $V(r) = \sqrt{r^2 - r^3}, r \in (-1,1)$.



Thank you for you attention!