EXISTENCE "IN THE LARGE" AND UNIQUENESS OF A SOLUTION TO PRIMITIVE EQUATIONS

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OVERVIEW

1. Problem formulation.
2. Auxiliary results.
3. Solution for 2D-case.
4. Peculiarities for 3D-case.
5. Primitive equations.

Leray Problem

Let $\mathbf{u} = (u_1, u_2, u_3)$ be the velocity vector and $p -$ the pressure. The system of Navier-Stokes equations describing flow of incompressible liquid in a bounded domain Ω is of the form

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p + u_k\mathbf{u}_{x_k} = \mathbf{f},$$

$$\operatorname{div}\mathbf{u} = 0, \qquad \mathbf{u}|_{\partial\Omega} = \mathbf{0}, \quad \mathbf{u}(\mathrm{x}, 0) = \mathbf{u}_0(\mathrm{x}); \tag{1}$$

hereafter the following notations are used

$$\partial_i = \frac{\partial}{\partial x_i}, \quad \|f\|_q = \|f\|_{L_q}, \quad \|f\| = \|f\|_{L_2},$$

$x = (x_1, x_2)$ or $x = (x_1, x_2, x_3)$,

$$u_k \mathbf{u}_{x_k} = (u_1 \partial_1 u_1 + u_2 \partial_2 u_1 + u_3 \partial_3 u_1, \ldots),$$

$$\|\mathbf{u}_x\|^2 = \sum_{i,j=1}^{3} \int_{\Omega} (\partial_i u_j)^2 dx.$$

The Leray problem is formulated as follows: *Let* Ω *be a bounded Lipschitz domain in 3D space. For any* $\nu > 0$ *(viscosity)* , $T > 0$ *(time interval) and arbitrary smooth* $\mathbf{f}$ *and* $\mathbf{u}_0$ *to prove existence and uniqueness of a solution* $\mathbf{u}(x,t) \in \mathbf{H}_0^1(\Omega)$ *and such that the norm* $\|\mathbf{u}_x(t)\|$ *is continuous in time on* $[0, T]$.

The following auxiliary results are used

Lemma 1. *For any* $f \in H_0^1(\Omega)$ *the following estimate holds in 2D case*

$$\|f\|_4^4 \leq 4\|\partial_1 f\|\,\|\partial_2 f\|\,\|f\|^2 \leq c_1\|\nabla f\|^2\,\|f\|^2. \quad (2)$$

Proof. Continue $f \in H_0^1$ on the whole plane R^2 by zero. From the relation

$$f^2(x) = 2\int_{-\infty}^{x_k} f f_{x_k} dx_k, \quad k = 1, 2,$$

it follows

$$\max_{x_k} f^2(x) \leq 2\int_{-\infty}^{\infty} |f f_{x_k}| dx_k, \quad k = 1, 2.$$

Then

$$\int\limits_{R^2} f^4(x)dx \leq \int\limits_{-\infty}^{\infty} \max_{x_2} f^2 dx_1 \int\limits_{-\infty}^{\infty} \max_{x_1} f^2 dx_2$$

$$\leq 4 \int\limits_{R^2} |f f_{x_2}| dx \int\limits_{R^2} |f f_{x_1}| dx \leq 4\|f\|^2 \|f_{x_1}\| \, \|f_{x_2}\|$$

$$\leq 2\|f\|^2 \|\nabla f\|^2.$$

Lemma 2. For any $f \in H_0^1(\Omega)$ the following estimate holds in 3D case

$$\|f\|_4^4 \leq 8\|\partial_1 f\| \, \|\partial_2 f\| \, \|\partial_3 f\| \, \|f\| \leq c_2 \|\nabla f\|^3 \, \|f\|. \tag{3}$$

Proof. As before,

$$\|f\|_4^4 \le 4 \int_{-\infty}^{\infty} \|f\|_{2,R^2}^2 \, \|f_{x_1}\|_{2,R^2} \, \|f_{x_2}\|_{2,R^2} \, dx_3$$

$$\le 4 \max_{x_3} \|f\|_{2,R^2}^2 \int_{-\infty}^{\infty} \|f_{x_1}\|_{2,R^2} \, \|f_{x_2}\|_{2,R^2} \, dx_3$$

$$\le 8 \int_{-\infty}^{\infty} |f f_{x_3}| dx_3 \|f_{x_1}\| \, \|f_{x_2}\|$$

$$\le 8 \|f_{x_1}\| \, \|f_{x_2}\| \, \|f_{x_3}\| \, \|f\| \le c_2 \|\nabla f\|^3 \, \|f\|.$$

Remark. Estimates (2) and (3) are severe, i.e. powers cannot be changed. The constants c_1 and c_2 do not depend on Ω and f.

From (2) and (3) and the Young inequality we have

$$\|f\|_4^4 \le \varepsilon \|\nabla f\|^4 + \varepsilon^{-1} \|f\|^4, \qquad \Omega \in R^2, \quad (4)$$

$$\|f\|_4^4 \leq \frac{2}{\sqrt{3}} \left[\varepsilon \|\nabla f\|^4 + (3\varepsilon)^{-1/3} \|f\|^4\right], \ \Omega \in R^3. \tag{5}$$

Lemma 3. *For any* $f \in H_0^1(\Omega)$, $\Omega \in R^3$, *the following inequality holds*

$$\|f\|_6 \leq (48)^{1/6} \|\nabla f\|. \tag{6}$$

Proof. For simplicity, assume $f \geq 0$. Then

$$\|f\|_6^6 = \int_\Omega f^6 dx = \int_{-\infty}^{\infty} dx_1 \int_{R^2} f^3 \cdot f^3 dx_2 dx_3$$

$$\leq \int_{-\infty}^{\infty} dx_1 \left[\int_{x_3} \max_{x_2} f^3 dx_3 \int_{x_2} \max_{x_3} f^3 dx_2 \right]$$

$$\leq 9 \int_{-\infty}^{\infty} dx_1 \left[\int_{R^2} f^2 f_{x_2} dx_2 dx_3 \int_{R^2} f^2 f_{x_3} dx_2 dx_3 \right]$$

$$\leq 9 \int_{-\infty}^{\infty} \left[\|f\|_{4,R^2}^4 \|f_{x_2}\|_{R^2} \|f_{x_3}\|_{R^2} \right] dx_1 .$$

From the above inequality one obtains

$$\|f\|_6^6 \le 9 \max_{x_1} \|f\|_{4,R^2}^4 \, \|f_{x_2}\|_{R^3} \, \|f_{x_3}\|_{R^3}$$

$$\le 36 \int\limits_{R^3} |f^3 f_{x_1}| dx \, \|f_{x_2}\|_{R^3} \, \|f_{x_3}\|_{R^3}$$

$$\le 36 \|f\|_6^3 \, \|f_{x_1}\| \, \|f_{x_2}\| \, \|f_{x_3}\|,$$

which yields

$$\|f\|_6 \le \left(36 \|f_{x_1}\| \, \|f_{x_2}\| \, \|f_{x_3}\|\right)^{1/3} \le (48)^{1/6} \|\nabla f\|,$$

Q.E.D.

Lemma 4 (The Gronwall inequality). *Let* $y(t) \ge 0$ *satisfy for almost all* $t \in [0, T]$ *the inequality*

$$y'(t) \le C_1(t) y(t) + C_2(t),$$

where $C_i(t)$ – *nonnegative integrable functions. Then*

$$y(t) \le \exp\left\{ \int\limits_0^t C_1(t) dt \right\} \left[y(0) + \int\limits_0^t C_2(t) dt \right]. \tag{7}$$

A PRIORI ESTIMATES

For simplicity we put $\mathbf{f} = \mathbf{0}$ and rewrite (1)

$$\mathbf{u}_t - \nu \Delta \mathbf{u} + \nabla p + u_k \mathbf{u}_{x_k} = \mathbf{0},$$

$$\operatorname{div} \mathbf{u} = 0, \qquad \mathbf{u}|_{\partial\Omega} = \mathbf{0}, \quad \mathbf{u}(\mathrm{x}, 0) = \mathbf{u}_0(\mathrm{x}). \tag{1}$$

1. Take a scalar product in $\mathbf{L}_2$ of (1) and $\mathbf{u}$:

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}\|^2 + \nu \|\mathbf{u}_x\|^2 = 0. \tag{8}$$

Integration in t gives

$$\max_t \|\mathbf{u}(t)\| \leq \|\mathbf{u}_0\| \equiv M, \tag{9}$$

$$2\nu \int_0^\infty \|\mathbf{u}_x(t)\|^2 dt \leq M^2. \tag{10}$$

2. From (8) and (9) we have

$$\nu \|\mathbf{u}_x\|^2 = (\mathbf{u}, \mathbf{u}_t) \le M \|\mathbf{u}_t\|. \qquad (11)$$

3. Differentiate (1) in t:

$$\mathbf{u}_{tt} - \nu \Delta \mathbf{u}_t + \nabla p_t + u_k \mathbf{u}_{tx_k} + u_{kt} \mathbf{u}_{x_k} = \mathbf{0},$$

$$\operatorname{div} \mathbf{u}_\mathrm{t} = 0, \qquad \mathbf{u}_\mathrm{t}|_{\partial\Omega} = \mathbf{0}. \qquad (12)$$

Take a scalar product of the first eq-n (12) and $\mathbf{u}_t$:

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}_t\|^2 + \nu \|\mathbf{u}_{tx}\|^2 + (u_{kt} \mathbf{u}_{x_k}, \mathbf{u}_t) = 0. \qquad (13)$$

Estimate the scalar product from (13). In 2D case we have

$$|(u_{kt}\mathbf{u}_{x_k}, \mathbf{u}_t)| = |(u_{kt}\mathbf{u}, \mathbf{u}_{tx_k})| \le \|\mathbf{u}_{tx}\|\,\|\mathbf{u}_t\|_4\|\mathbf{u}\|_4$$

$$\le c\|\mathbf{u}_{tx}\|^{3/2}\,\|\mathbf{u}_t\|^{1/2}\|\mathbf{u}_x\|^{1/2}\|\mathbf{u}\|^{1/2}$$

$$\le c\sqrt{M}\|\mathbf{u}_{tx}\|^{3/2}\,\|\mathbf{u}_t\|^{1/2}\|\mathbf{u}_x\|^{1/2}$$

$$\le \frac{\nu}{2}\|\mathbf{u}_{tx}\|^2 + cM^2/\nu^3\|\mathbf{u}_x\|^2\|\mathbf{u}_t\|^2.$$

Then from (13) one gets

$$\frac{d}{dt}\|\mathbf{u}_t\|^2 + \nu\|\mathbf{u}_{tx}\|^2 \le cM^2/\nu^3\|\mathbf{u}_x\|^2\|\mathbf{u}_t\|^2. \tag{14}$$

Using the Gronwall inequality (7), from (14) we obtain

$$\max_{0\le t\le T}\|\mathbf{u}_t(t)\|^2 \le \|\mathbf{u}_t(0)\|^2 e^{cM^2/\nu^3\int_0^\infty\|\mathbf{u}_x\|^2dt}$$

$$\le \|\mathbf{u}_t(0)\|^2 e^{cM^4/\nu^4}. \tag{15}$$

It is easy to see that the norm $\|\mathbf{u}_t(0)\|$ can be estimated from above by some norm of the initial condition $\mathbf{u}_0$. Therefore, for 2D case we have the final a priori eqtimate

$$\max_{0 \le t \le T} \|\mathbf{u}_t(t)\| \le c_T, \tag{16}$$

where c_T depends on the time interval, the norm of the initial condition, and ν.

Using (16) and the ordinary technique, it is not difficult to prove existence and uniqueness of a solution from the Sobolev space $\mathbf{H}^1(Q_T)$, where $Q_T = \Omega \times (0, T)$. From this it follows that the norm of this solution $\|\mathbf{u}_x\|$ is continuous in time.

3D case. The estimates (9)–(11) are valid. As for estimation of the scalar product from

(13), in 3D case we have

$$|(u_{kt}\mathbf{u}_{x_k}, \mathbf{u}_t)| = |(u_{kt}\mathbf{u}, \mathbf{u}_{tx_k})| \leq \|\mathbf{u}_{tx}\| \, \|\mathbf{u}_t\|_4 \|\mathbf{u}\|_4$$

$$\leq c\|\mathbf{u}_{tx}\|^{7/4} \, \|\mathbf{u}_t\|^{1/4} \|\mathbf{u}_x\|^{3/4} \|\mathbf{u}\|^{1/4}$$

$$\leq cM^{1/4} \|\mathbf{u}_{tx}\|^{7/4} \, \|\mathbf{u}_t\|^{1/4} \|\mathbf{u}_x\|^{3/4}$$

$$\leq \frac{\nu}{2} \|\mathbf{u}_{tx}\|^2 + c(M, \nu) \|\mathbf{u}_x\|^6 \|\mathbf{u}_t\|^2.$$