Existence and Uniqueness of a Solution to the Primitive Equations with Stratification "in the large"

G.M. Kobelkov
Moscow State University,
Inst. of Numerical Maths. RAS

Let $\mathbf{u} = (u_1, u_2, u_3)$, $\hat{\mathbf{u}} = (u_1, u_2)$. Indices i, j range 1, 2, and k — from 1 to 3. Space variables are $x = (x_1, x_2, x_3)$ or x, y, z; $x' = (x_1, x_2)$.

Let $\Omega = \Omega' \times [0, 1]$, where Ω' is a 2D domain with a piecewise smooth boundary $\partial\Omega'$; $\partial\Omega = S \cup S_1$, where S is a lateral part of $\partial\Omega$.

The system of primitive equations is

$$\hat{\mathbf{u}}_t - \nu\Delta'\hat{\mathbf{u}} - \nu_v\hat{\mathbf{u}}_{zz} + l\hat{\mathbf{u}} + \nabla' p + u_k\hat{\mathbf{u}}_{x_k} = \mathbf{f},$$

$$\frac{\partial p}{\partial x_3} = -g\rho,$$

$$\operatorname{div}\mathbf{u} = \varphi, \quad \rho_t - \mu\Delta'\rho - \partial_3(\mu_v(\rho_z)\partial_3\rho) + u_k\rho_{x_k} = 0,$$

$$\hat{\mathbf{u}}\cdot\mathbf{n} = \frac{\partial\hat{\mathbf{u}}}{\partial n}\times\mathbf{n} = 0 \quad \text{on} \quad S\times[0,T],$$

$$u_3 = 0,\ \partial_3 u_1 = \partial_3 u_2 = 0 \quad \text{on} \quad S_1\times[0,T],$$

$$\left.\frac{\partial\rho}{\partial n}\right|_{\partial\Omega} = 0,\ \hat{\mathbf{u}}(t,\, x) = \hat{\mathbf{u}}_0(x),$$

$$\int_0^1 \operatorname{div}'\hat{\mathbf{u}}_0 dz = 0,\ \rho(x,0) = \rho_0(x);\ \int_\Omega \varphi dx = 0; \tag{1}$$

we assume summation over repeating indices in products; g is acceleration of gravity, $\mathbf{n}$ is an outer unit normal to the boundary, $\mathbf{a}\times\mathbf{b} = a_1b_2 - a_2b_1$. The operator $l\hat{\mathbf{u}}$ is of the form

$$l\hat{\mathbf{u}} = \omega(u_2, -u_1).$$

Stratification means that the viscosity coefficient μ_v depends on ρ_z. We impose the following conditions on μ_v:

1) $\mu_v(s)$ is bounded, smooth and non-increasing for $s \leq 0$;
2) $\mu_v(s)$ is constant for $s \geq 0$.

The problem consists in proving existence and uniqueness of a solution to (1) "in the large", i.e. the norm $\|\widehat{\mathbf{u}}_x\|$ has to be continuous in time on any finite time interval $[0,T]$ without assumptions on smallness of initial data, viscosity coefficients or size of a domain.

A priori estimates. Multiply the equation

$$\rho_t - \mu\Delta'\rho - (\mu_v\rho_z)_z + u_k\rho_{x_k} = 0$$

by ρ^3 and integrate over Ω. After simple estimation we get

$$\max_{0\le t\le T} \|\rho(t)\|_4 \le c\|\rho_0\|_4. \tag{2}$$

From the second equation of (1)

$$\frac{\partial p}{\partial x_3} = -g\rho$$

and (2) it follows

$$\max_{0\le t\le T} \|\partial_3 p(t)\|_4 \le c\|\rho_0\|_4. \tag{3}$$

Multiply the first equation of (1)

$$\hat{\mathbf{u}}_t - \nu\Delta'\hat{\mathbf{u}} - \nu_v\hat{\mathbf{u}}_{zz} + l\hat{\mathbf{u}} + \nabla' p + u_k\hat{\mathbf{u}}_{x_k} = \mathbf{f},$$

by $\hat{\mathbf{u}}$ and integrate over Ω:

$$\frac{1}{2}\frac{d}{dt}\|\hat{\mathbf{u}}\|^2 + \nu\|\hat{\mathbf{u}}'_x\|^2 + \nu_v\|\hat{\mathbf{u}}_z\|^2 - (p, \mathrm{div}\,'\hat{\mathbf{u}}) = (\mathbf{f}, \hat{\mathbf{u}}). \tag{4}$$

Estimate the scalar product from the left-hand side of (4). From the incompressibility equation and boundary conditions it follows

$$\|u_3\| \le c\|\partial_3 u_3\| \le c\|\hat{\mathbf{u}}_{x'}\|. \tag{5}$$

Then

$$|(p, \mathrm{div}\,'\hat{\mathbf{u}})| = |(p, \partial_3 u_3)| = |(\partial_3 p, u_3)|$$

$$\le c\|\partial_3 p\|\,\|\hat{\mathbf{u}}_{x'}\| \le \frac{\nu}{2}\|\hat{\mathbf{u}}_{x'}\|^2 + \frac{c}{\nu}\|\partial_3 p\|^2.$$

So

$$\frac{d}{dt}\|\hat{\mathbf{u}}\|^2 + c_1\|\hat{\mathbf{u}}_x\|^2 \le c_2\left(\|\partial_3 p\|^2 + \|\mathbf{f}\|^2_{-1}\right).$$

Integrating this inequality in t from 0 to T and taking into account (3), we get

$$\max_{0\le t\le T} \|\hat{\mathbf{u}}(t)\|^2 + \int_0^T \|\hat{\mathbf{u}}_x\|^2 dt \le c. \qquad (6)$$

For u_3 from (5) and (6):

$$\int_0^T \left(\|u_3\|^2 + \|\partial_3 u_3\|^2\right) dt \le c. \qquad (7)$$

Estimation of the norm $\|p\|_4$. Represent p as $p = p_1 + p_2$, where $p_1(t,x,y) = \int_0^1 p(t,x,y,z)dz$. Then p_2 is an antiderivative of $\partial_3 p$ in z and $\int_0^1 p_2 dz = 0$. Since p is determined from (1) up to a constant, in what follows we shall assume $\int_{\Omega'} p_1^3 dx' = 0$. Then,

$$\|p_2\|_4 \le c\|\partial_3 p_2\|_4 \le c. \qquad (8)$$

Estimate the norm of p_1. Take the scalar product of the first equation and $\nabla'(\Delta')^{-1}p_1^3$; after some estimations we have

$$\|p_1\|_4 \le c(\|v_x\|^{1/2} + \|v\|^{1/2} + 1)\|v\|^{1/2},\ v = \hat{\mathbf{u}}^2. \qquad (9)$$

Here we used the inequality

$$|v|_4 \le c\left(|v_{x'}|^{1/2} + |v|^{1/2}\right)|v|^{1/2} \tag{10}$$

being valid for 2D case and $|\cdot|$ means the norm in x, y variables. As for estimation of the integral over $\partial\Omega$, we differentiate the first b.c. in the tangent direction and use the second b.c. So, this integral is reduced to the one containing derivatives of the normal components. It gives possibility to estimate this integral properly.

Estimation of the norm $\max_t \|\hat{\mathbf{u}}\|_4$. Multiply (1) by $\hat{\mathbf{u}}^2\hat{\mathbf{u}}$ and integrate over Ω:

$$\frac{1}{4}\,\frac{d}{dt}\,\|\hat{\mathbf{u}}(t)\|_4^4 - \nu(\mathbf{\Delta}'\hat{\mathbf{u}}, \hat{\mathbf{u}}^2\hat{\mathbf{u}}) - \nu_v(\hat{\mathbf{u}}_{zz}, \hat{\mathbf{u}}^2\hat{\mathbf{u}})$$

$$+(\nabla' p, \hat{\mathbf{u}}^2\hat{\mathbf{u}}) + (u_k\hat{\mathbf{u}}_{x_k}, \hat{\mathbf{u}}^2\hat{\mathbf{u}}) = 0. \tag{11}$$

The last scalar product in (11) equals zero.

Then,

$$-\nu(\mathbf{\Delta}'\hat{\mathbf{u}}, |\hat{\mathbf{u}}|^2\hat{\mathbf{u}}) - \nu_v(\hat{\mathbf{u}}_{zz}, \hat{\mathbf{u}}^2\hat{\mathbf{u}}) \geq$$

$$c\left(\int_\Omega v\,|\nabla\hat{\mathbf{u}}|^2 dx + \|v_x\|^2\right),$$

so (11) may be rewritten as

$$\frac{d}{dt}\|v(t)\|^2 + c_1\left(\int_\Omega v\,|\nabla\hat{\mathbf{u}}|^2 dx + \|v_x\|^2\right) \leq c_2|(\nabla' p, \hat{\mathbf{u}}^2\hat{\mathbf{u}})| \tag{12}$$

Using the estimates for p_1 and p_2, it is possible to estimate the scalar product of (12):

$$|(\nabla' p, \hat{\mathbf{u}}^2\hat{\mathbf{u}})| = |(p, \mathrm{div}'(\hat{\mathbf{u}}^2\hat{\mathbf{u}}))| \leq |(p\hat{\mathbf{u}}, \nabla'\hat{\mathbf{u}}^2)|$$

$$+|(p\hat{\mathbf{u}}^2, \mathrm{div}'\hat{\mathbf{u}})| \leq c(|p|v, |\hat{\mathbf{u}}_x|)$$

$$\leq c((|p_1|v, |\hat{\mathbf{u}}_x|) + (|p_2|v, |\hat{\mathbf{u}}_x|)). \tag{13}$$

Estimate scalar products of (13) separately:

$$(|p_1|v, |\hat{\mathbf{u}}_x|) \leq \frac{\nu}{4}\|v_x\|^2 + \frac{c}{\nu}\|v\|^2\|\hat{\mathbf{u}}_x\|^2$$

$$+c\|v\|^2\|\hat{\mathbf{u}}_x\| + c\|v\|\|\hat{\mathbf{u}}_x\|, \tag{14}$$

$$(|p_2|v, |\hat{\mathbf{u}}_x|) \leq \int_0^1 |p_2|_4|v|_4|\hat{\mathbf{u}}_x|dz = |p_2|_4 \int_0^1 |v|_4|\hat{\mathbf{u}}_x|dz$$

$$\leq c\|v_{x'}\|^{1/2}\|v\|^{1/2}\|\hat{\mathbf{u}}_x\| \leq \frac{\nu}{2}\|v_x\|^2 + c\|v\|^2 + c\|\hat{\mathbf{u}}_x\|^2. \tag{15}$$

So, from (12) we have

$$\frac{d}{dt}\|v(t)\|^2 + c\int_\Omega \hat{\mathbf{u}}^2|\hat{\mathbf{u}}_x|^2 dx \leq c\|v\|^2(\|\hat{\mathbf{u}}_x\|^2 + \|\hat{\mathbf{u}}_x\|). \tag{16}$$

From Gronwall's inequality and (16), it follows

$$\max_{0\leq t\leq T} \|\hat{\mathbf{u}}(t)\|_4 \leq c. \tag{17}$$

In turn, (16) and (17) yield

$$\int_0^T \int_\Omega \hat{\mathbf{u}}^2|\hat{\mathbf{u}}_x|^2 dxdt \leq c. \tag{18}$$

Since $\|u_j\partial_3 u_3\| = \|u_j(\partial_1 u_1 + \partial_2 u_2)\|$, then from

(18) we have

$$\int_0^T \int_\Omega u_j^2(\partial_3 u_3)^2 dxdt \le c, \quad j = 1, 2. \tag{19}$$

Differentiate equations (1) in z. Denote $\mathbf{u}_z = \mathbf{v}$, $\hat{\mathbf{v}} = (v_1, v_2)$; so

$$\hat{\mathbf{v}}_t - \nu\Delta'\hat{\mathbf{v}} - \nu_v\hat{\mathbf{v}}_{zz} + \nabla' p_z + v_k\hat{\mathbf{u}}_{x_k} + u_k\hat{\mathbf{v}}_{x_k} = \mathbf{0},$$
$$\operatorname{div}\mathbf{v} = 0. \tag{20}$$

Multiply the first equation of (20) by $\hat{\mathbf{v}}$ and integrate over Ω:

$$\frac{1}{2}\frac{d}{dt}\|\hat{\mathbf{v}}\|^2 + \nu\|\hat{\mathbf{v}}_x\|^2 - (p_z, \operatorname{div}'\hat{\mathbf{v}})$$
$$+(v_k\hat{\mathbf{u}}_{x_k}, \hat{\mathbf{v}}) + (u_k\hat{\mathbf{v}}_{x_k}, \hat{\mathbf{v}}) = 0. \tag{21}$$

The last scalar product equals zero. Estimate the two remaining scalar products in (21):

$$|(p_z, \operatorname{div}'\hat{\mathbf{v}})| \le \|p_z\|\,\|\operatorname{div}'\hat{\mathbf{v}}\| \le c\|\operatorname{div}'\hat{\mathbf{v}}\| \le \frac{\nu}{4}\|\hat{\mathbf{v}}_x\|^2 + c.$$

Further,

$$|(v_k \hat{\mathbf{u}}_{x_k}, \hat{\mathbf{v}})| = |(v_k \hat{\mathbf{u}}, \hat{\mathbf{v}}_{x_k})| \leq |(v_j \hat{\mathbf{u}}, \hat{\mathbf{v}}_{x_j})| + |(v_3 \hat{\mathbf{u}}, \hat{\mathbf{v}}_{x_3})|$$

$$\leq c\|\hat{\mathbf{u}}\|_4 \|\hat{\mathbf{v}}\|_4 \|\hat{\mathbf{v}}_x\| + |(\mathrm{div}'\hat{\mathbf{u}} \cdot \hat{\mathbf{u}}, \hat{\mathbf{v}}_{x_3})| \leq c\|\hat{\mathbf{v}}_x\|^{7/4}\|\hat{\mathbf{v}}\|^{1/4}$$

$$+\|\mathrm{div}'\hat{\mathbf{u}} \cdot \hat{\mathbf{u}}\| \, \|\hat{\mathbf{v}}_x\| \leq \frac{\nu}{4}\|\hat{\mathbf{v}}_x\|^2 + c\|\hat{\mathbf{v}}\|^2 + c\|\mathrm{div}'\hat{\mathbf{u}} \cdot \hat{\mathbf{u}}\|^2.$$

So, from (21) it follows

$$\frac{d}{dt}\|\hat{\mathbf{v}}\|^2 + c_1\|\hat{\mathbf{v}}_x\|^2 \leq c_2\left(\|\hat{\mathbf{v}}\|^2 + \|\mathrm{div}'\hat{\mathbf{u}} \cdot \hat{\mathbf{u}}\|^2\right). \tag{22}$$

From (22) and Gronwall's inequality one obtains

$$\max_{0\leq t\leq T}\|\hat{\mathbf{v}}(t)\|^2 + \nu\int_0^T \|\hat{\mathbf{v}}_x\|^2 dt \leq c. \tag{23}$$

Multiply (20) by $\hat{\mathbf{v}}^2\hat{\mathbf{v}}$ and integrate over:

$$\frac{1}{4}\frac{d}{dt}\|\hat{\mathbf{v}}\|_4^4+\nu\int_\Omega\left(\hat{\mathbf{v}}^2(\hat{\mathbf{v}}_{x'})^2\frac{1}{2}[(\hat{\mathbf{v}}^2)_x]^2\right)dx$$

$$+\nu_v\int_\Omega\left(\hat{\mathbf{v}}^2(\hat{\mathbf{v}}_z)^2+\frac{1}{2}[(\hat{\mathbf{v}}^2)_z]^2\right)dx$$

$$-(p_z,\mathrm{div}'(\hat{\mathbf{v}}^2\hat{\mathbf{v}}))+(v_k\hat{\mathbf{u}}_{x_k},\hat{\mathbf{v}}^2\hat{\mathbf{v}})=0. \tag{24}$$

Estimate the scalar products in (24):

$$I_1=|(p_z,\mathrm{div}'(\hat{\mathbf{v}}^2\hat{\mathbf{v}}))|\le|(p_z,\hat{\mathbf{v}}^2\mathrm{div}'\hat{\mathbf{v}})|+|(p_z,\hat{\mathbf{v}}\nabla'\hat{\mathbf{v}}^2)| \tag{25}$$

The first term is estimated with the use of the Hölder inequality for the functions p_z, $|\hat{\mathbf{v}}|\mathrm{div}'\hat{\mathbf{v}}$ and $|\hat{\mathbf{v}}|$ with the powers 4, 2 and 4. Taking into account the boundedness of the norms of p_z in L_4 in t and the estimate $\|w\|_4\le c\|w_x\|^{1/2}\|w\|^{1/2}$ being valid for 2D case, we obtain

$$|(p_z,\hat{\mathbf{v}}^2\mathrm{div}'\hat{\mathbf{v}})|\le\|p_z\|_4\,\|\hat{\mathbf{v}}\cdot\mathrm{div}'\hat{\mathbf{v}}\|\,\|\hat{\mathbf{v}}\|_4$$

$$\le\varepsilon\int_\Omega\hat{\mathbf{v}}^2(\hat{\mathbf{v}}_x)^2dx+c_\varepsilon\left(\|\hat{\mathbf{v}}_x\|^2+\|\hat{\mathbf{v}}\|^2\right).$$

The second term is estimated in a similar way. So

$$|(p_z, \mathrm{div}'(\hat{\mathbf{v}}^2\hat{\mathbf{v}}))| \le \varepsilon \int_\Omega \hat{\mathbf{v}}^2(\hat{\mathbf{v}}_x)^2 dx + c_\varepsilon \left(\|\hat{\mathbf{v}}_x\|^2 + \|\hat{\mathbf{v}}\|^2\right) \tag{26}$$

Estimate the second scalar product of (24):

$$I_2 = |(v_k \hat{\mathbf{u}}_{x_k}, \hat{\mathbf{v}}^2\hat{\mathbf{v}})| \le |(v_j \hat{\mathbf{u}}_{x_j}, \hat{\mathbf{v}}^2\hat{\mathbf{v}})| + |(v_3 \hat{\mathbf{u}}_{x_3}, \hat{\mathbf{v}}^2\hat{\mathbf{v}})|$$

$$\equiv I_2' + I_2''.$$

The first scalar product is estimated as

$$|(v_j \hat{\mathbf{u}}_{x_j}, \hat{\mathbf{v}}^2\hat{\mathbf{v}})| \le |(\mathrm{div}'\hat{\mathbf{v}} \cdot \hat{\mathbf{v}}, \hat{\mathbf{v}}^2\hat{\mathbf{u}})|$$

$$+|(v_j \hat{\mathbf{u}}, \hat{\mathbf{v}}\hat{\mathbf{v}}_{x_j}^2)| + |(v_j \hat{\mathbf{u}}, \hat{\mathbf{v}}^2\hat{\mathbf{v}}_{x_j})|.$$

Estimate, e.g. the third scalar product. Use the Hölder inequality for the functions $v_j\hat{\mathbf{v}}_{x_j}$, $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}^2$ with the powers 2, 4 and 4:

$$I_2' = |(v_j \hat{\mathbf{u}}, \hat{\mathbf{v}}^2\hat{\mathbf{v}}_{x_j})| \le c\|\hat{\mathbf{v}}\cdot\hat{\mathbf{v}}_x\| \, \|\hat{\mathbf{u}}\|_4 \left(\int_\Omega |\hat{\mathbf{v}}|^8 dx\right)^{1/4}.$$

Taking into account the boundedness of $\|\hat{\mathbf{u}}\|_4$ in t and the estimate $\|w\|_4 \le c\|w_x\|^{3/4}\|w\|^{1/4}$ being hold in 3D case, one obtains

$$I_2' \le c\|\hat{\mathbf{v}} \cdot \hat{\mathbf{v}}_x\| \, \|w_x\|^{3/4}\|w\|^{1/4},$$

where $w = \hat{\mathbf{v}}^2$. From this inequality we have

$$I_2' \le \varepsilon \int_\Omega \hat{\mathbf{v}}^2(\hat{\mathbf{v}}_x)^2 dx + \varepsilon \int_\Omega [(\hat{\mathbf{v}}^2)_x]^2 dx + c_\varepsilon \|\hat{\mathbf{v}}\|_4^4. \tag{27}$$

Estimate I_2'':

$$I_2'' = |(v_3 \hat{\mathbf{u}}_{x_3}, \hat{\mathbf{v}}^2\hat{\mathbf{v}})| = |(\mathrm{div}\,'\hat{\mathbf{u}} \cdot \hat{\mathbf{v}}, \hat{\mathbf{v}}^2\hat{\mathbf{v}})| = |(\hat{\mathbf{u}} \cdot \hat{\mathbf{v}}^2, \nabla'\hat{\mathbf{v}}^2)|$$

$$\le \|\hat{\mathbf{u}}\|_4 \, \|\hat{\mathbf{v}}^2\|_4 \, \|\nabla'\hat{\mathbf{v}}^2\| \le c\|w_x\|^{3/4} \, \|w\|^{1/4} \, \|\nabla'\hat{\mathbf{v}}^2\|$$

$$\le \varepsilon \int_\Omega \hat{\mathbf{v}}^2(\hat{\mathbf{v}}_x)^2 dx + \varepsilon \int_\Omega [(\hat{\mathbf{v}}^2)_x]^2 dx + c_\varepsilon \|\hat{\mathbf{v}}\|_4^4. \tag{28}$$

Thus, using (26) – (28), from (26) we have

$$\frac{d}{dt}\|\hat{\mathbf{v}}\|_4^4 + c_1 \int_\Omega \left(\hat{\mathbf{v}}^2(\hat{\mathbf{v}}_x)^2 + \frac{1}{2}[(\hat{\mathbf{v}}^2)_x]^2\right) dx$$

$$\le c_2 \left(\|\hat{\mathbf{v}}_x\|^2 + \|\hat{\mathbf{v}}\|^2 + \|\hat{\mathbf{v}}\|_4^4\right)$$

from what follows

$$\max_t \|\hat{\mathbf{v}}\|_4^4 + \int_0^T \int_\Omega \left(\hat{\mathbf{v}}^2(\hat{\mathbf{v}}_x)^2 + [(\hat{\mathbf{v}}^2)_x]^2\right) dxdt \le c. \tag{29}$$

All the above inequalities yield

$$\max_t \|\hat{\mathbf{u}}\|_4^4 + \max_t \|\hat{\mathbf{u}}_z\|_4^4$$

$$+\nu \int_0^T \int_\Omega \left(\hat{\mathbf{u}}^2\hat{\mathbf{u}}_x^2 + \hat{\mathbf{u}}_x^2 + u_{3z}^2 + \hat{\mathbf{u}}^2 u_{3z}^2\right) dxdt$$

$$+\nu \int_0^T \int_\Omega \left(\hat{\mathbf{u}}_z^2\hat{\mathbf{u}}_{zx}^2 + \hat{\mathbf{u}}_{zx}^2 + u_{3zz}^2 + \hat{\mathbf{u}}_z^2 u_{3zz}^2\right) dxdt \le c. \tag{30}$$

Let us now obtain the estimate for ρ_z. Differentiation of the equation for ρ in z gives

$$\rho_{zt} - \mu\Delta'\rho_z - (\mu_v\rho_z)_{zz} + \mathbf{u}_z\cdot\nabla\rho + \mathbf{u}\cdot\nabla\rho_z = 0. \tag{31}$$

Scalar multiplication of (31) by ρ_z gives

$$\frac{1}{2}\frac{d}{dt}\|\rho_z\|^2 + \mu\|\rho_{zx'}\|^2 + (\mu_v\rho_{zz}, \rho_{zz})$$
$$+(\mu_v'\rho_{zz}, \rho_{zz}\rho_z) + (\mathbf{u}_z\cdot\nabla\rho, \rho_z) = 0. \tag{32}$$

Due to the properties imposed on μ_v, we have $(\mu'_v \rho_{zz}, \rho_{zz}\rho_z) \geq 0$.

Estimation of the last scalar product gives

$$|(\mathbf{u}_z \cdot \nabla\rho, \rho_z)| \leq \varepsilon\|\rho_{zx}\|^2 + c_\varepsilon(\|\hat{\mathbf{v}}_x\|^2 + \|\rho_x\|^2) + c \tag{33}$$

Then, from (32) it follows

$$\max_{0\leq t\leq T} \|\rho_z\|^2 + \int_0^T \|\rho_{zx}\|^2 dt \leq c \tag{34}$$

Differentiate (1) in t:

$$\hat{\mathbf{u}}_{tt} - \nu\boldsymbol{\Delta}\hat{\mathbf{u}}_t + l\hat{\mathbf{u}}_t + \nabla' p_t + u_{kt}\hat{\mathbf{u}}_{x_k} + u_k\hat{\mathbf{u}}_{tx_k} = \mathbf{0},$$

$$\operatorname{div}\mathbf{u}_t = 0,$$

$$\rho_{tt} - \mu\Delta'\rho_t - (\mu_v\rho_{tz})_z - (\mu'_v\rho_{zt}\rho_z)_z$$

$$+u_{kt}\rho_{x_k} + u_k\rho_{tx_k} = 0. \tag{35}$$

Take the scalar product of the first equation of (35) and $\hat{\mathbf{u}}_t$, and the third equation of (35) and ρ_t. One has

$$\frac{1}{2}\,\frac{d}{dt}\,\|\hat{\mathbf{u}}_t\|^2 + \nu\|\hat{\mathbf{u}}_{tx'}\|^2 + \nu_v\|\hat{\mathbf{u}}_{tz}\|^2 + (\nabla' p_t, \hat{\mathbf{u}}_t)$$

$$+(u_{kt}\hat{\mathbf{u}}_{x_k}, \hat{\mathbf{u}}_t) = 0,$$

$$\frac{1}{2}\,\frac{d}{dt}\,\|\rho_t\|^2 + \mu\|\rho_{tx'}\|^2 + (\mu_v, \rho_{tz}^2)$$

$$+(\mu_v'\rho_z, \rho_{tz}^2) + (u_{kt}\rho_{x_k}, \rho_t) = 0. \tag{36}$$

The scalar products in (36) may be estimated as

$$|(\nabla' p_t, \hat{\mathbf{u}}_t)| \le \varepsilon\|\hat{\mathbf{u}}_{tx}\|^2 + c_1\|\rho_t\|^2;$$

$$|(u_{kt}\hat{\mathbf{u}}_{x_k}, \hat{\mathbf{u}}_t)| \le \varepsilon\|\hat{\mathbf{u}}_{tx}\|^2 + c\|\hat{\mathbf{u}}_t\|^2.$$

Further,

$$|(u_{kt}\rho_{x_k},\rho_t)| = |(u_{kt}\rho,\rho_{tx_k})| \le |(u_{jt}\rho,\rho_{tx_j})| + |(u_{3t}\rho,\rho_{tx_3})|$$

$$|(u_{jt}\rho,\rho_{tx_j})| \le c\|\widehat{\mathbf{u}}_t\|_4\,\|\rho\|_4\,\|\rho_{tx}\| \le c\|\widehat{\mathbf{u}}_t\|_4\,\|\rho_{tx}\|$$

$$\le \varepsilon\|\rho_{tx}\|^2 + c\|\widehat{\mathbf{u}}_t\|_4^2 \le \varepsilon\|\rho_{tx}\|^2 + \varepsilon\|\widehat{\mathbf{u}}_{tx}\|^2 + c\|\widehat{\mathbf{u}}_t\|^2;$$

$$|(u_{3t}\rho,\rho_{tx_3})| \le |(u_{3tz}\rho,\rho_t)| + |(u_{3t}\rho_z,\rho_t)| \equiv I_3 + I_4;$$

$$I_3 \le \|u_{3tz}\|\,\|\rho\|_4\,\|\rho_t\|_4 \le c\|\widehat{\mathbf{u}}_{tx}\|\,\|\rho_t\|_4 \le \varepsilon\|\widehat{\mathbf{u}}_{tx}\|^2 + c\|\rho_t\|_4^2$$

$$\le \varepsilon\|\widehat{\mathbf{u}}_{tx}\|^2 + \varepsilon\|\rho_{tx}\|^2 + c\|\rho_t\|^2;$$

$$I_4 = |(u_{3t}\rho_z,\rho_t)| = \left|\int_0^1 \int_{\Omega'} \left(\int_0^{x_3} \operatorname{div}'\widehat{\mathbf{u}}_t dz\right) \rho_{x_3}\rho_t\, dx' dx_3\right|$$

$$\le \int_0^1 \left(\int_{\Omega'} \int_0^1 |\operatorname{div}'\widehat{\mathbf{u}}_t| dz \cdot |\rho_{x_3}|\,|\rho_t|\, dx'\right) dx_3.$$

Denote $\int_0^1 |\operatorname{div}'\widehat{\mathbf{u}}_t| dz = q(x')$. Then, to estimate I_4 we have

$$I_4 \leq \int_0^1 \left(\int_{\Omega'} q \, |\rho_{x_3}| \, |\rho_t| \, dx' \right) dx_3 \leq \int_0^1 |q| \, |\rho_z|_4 \, |\rho_t|_4 dz$$

$$\leq c|q| \int_0^1 |\rho_{zx'}|^{1/2} \, |\rho_z|^{1/2} \, |\rho_{tx'}|^{1/2} \, |\rho_t|^{1/2} dz.$$

Applying the Cauchy inequality and taking into account the estimate

$$|q|^2 = \int_{\Omega'} q^2 dx' = \int_{\Omega'} \left(\int_0^1 |\mathrm{div}\,' \hat{\mathbf{u}}_t| dz \right)^2 dx'$$

$$\leq \int_{\Omega'} \int_0^1 |\mathrm{div}\,' \hat{\mathbf{u}}_t|^2 dz dx' \leq c \|\hat{\mathbf{u}}_{tx}\|^2,$$

we have

$$I_4 \leq c \|\hat{\mathbf{u}}_{tx}\| \, \|\rho_{zx}\|^{1/2} \, \|\rho_z\|^{1/2} \, \|\rho_{tx}\|^{1/2} \, \|\rho_t\|^{1/2}$$

$$\leq \varepsilon \|\hat{\mathbf{u}}_{tx}\|^2 + \varepsilon \|\rho_{tx}\|^2 + c \|\rho_{zx}\|^2 \, \|\rho_t\|^2.$$

Choose sufficiently small ε. Then

$$\frac{d}{dt}\left(\|\hat{\mathbf{u}}_t\|^2+\|\rho_t\|^2\right)+c_1\|\hat{\mathbf{u}}_{tx}\|^2$$
$$+c_2\|\rho_{tx}\|^2-c_3\|\rho_t\|^2 \tag{37}$$
$$-c_4\|\hat{\mathbf{u}}_t\|^2-c_5\|\rho_{zx}\|^2\,\|\rho_t\|^2\le 0,$$

from what follows

$$\max_{0\le t\le T}\left(\|\hat{\mathbf{u}}_t(t)\|+\|\rho_t(t)\|\right)+\int_0^T\left(\|\hat{\mathbf{u}}_{tx}\|^2+\|\rho_{tx}\|^2\right)dt$$
$$\le c_T\left(\|\hat{\mathbf{u}}_t(0)\|^2+\|\rho_t(0)\|^2\right). \tag{38}$$

The right-hand side of (38) may be estimated by some constant depending on the norms $\|\hat{\mathbf{u}}_0\|_{\mathbf{W}_3^2}$, $\|\rho_0\|_{W_2^2}$. Thus, the final a priori estimate is of the form

$$\max_{0\le t\le T}(\|\rho\|_4+\|\rho_z\|+\|\hat{\mathbf{u}}_x\|+\|u_3\|+\|\hat{\mathbf{u}}_z\|_4+\|\hat{\mathbf{u}}_t\|$$

$$+\|\rho_t\|)+\int_0^T\left(\|\rho_x\|^2+\|\rho_{zx}\|^2+\|\rho_{tx}\|^2+\|\hat{\mathbf{u}}_x\|^2\right.$$

$$\left.+\|\hat{\mathbf{u}}_{zx}\|^2+\|\hat{\mathbf{u}}_{tx}\|^2+\|\hat{\mathbf{u}}\hat{\mathbf{u}}_x\|^2+\|\hat{\mathbf{u}}_z\hat{\mathbf{u}}_{zx}\|^2\right)dt$$

$$\le c_T\left(\|\hat{\mathbf{u}}_0\|^4_{W^2_3}+\|\rho_0\|^2_{W^2_2}\right). \qquad (39)$$

Let us now proceed to the proof of existence of a solution.

Introduce the spaces

- $\mathbf{V}_2$ — the space of vector functions $\hat{\mathbf{v}} = (v_1, v_2)$ from $\mathbf{W}_2^1(Q_T)$, satisfying boundary conditions and such that $\hat{\mathbf{v}}_z \in \mathbf{W}_2^1(Q_T)$ and $\int_0^1 \mathrm{div}'\hat{\mathbf{v}}(x', z, t)\, dz = 0$;

- R — the space of functions $r \in W_2^1(Q_T)$ such that $r_z \in W_2^1(Q_T)$.

The weak form of the density equation is

$$\int_{Q_t} (-\rho r_t + \nu_1 \rho_x r_x - u_k \rho r_{x_k})\, dxdt + \int_\Omega \rho r|_{t=t}\, dx - \int_\Omega \rho_0 r|_{t=0}\, dx = 0. \tag{40}$$

The weak form of the motion equation is

$$\int_{Q_t}\Big(-\hat{\mathbf{u}}\hat{\mathbf{v}}_t+\nu\hat{\mathbf{u}}_x\hat{\mathbf{v}}_x+l\hat{\mathbf{u}}\hat{\mathbf{v}}+\nabla'p\cdot\hat{\mathbf{v}}+u_k\hat{\mathbf{u}}_{x_k}\hat{\mathbf{v}}\Big)\,dxdt$$

$$+\int_\Omega \hat{\mathbf{u}}\hat{\mathbf{v}}|_{t=t}dx-\int_\Omega \hat{\mathbf{u}}_0\hat{\mathbf{v}}|_{t=0}dx=0; \tag{41}$$

here u_3 is uniquely determined from the relations $\operatorname{div}\mathbf{u}=0$, $u_3(t,x',0)=0$.

It is possible to eliminate p and u_3 from (41)

$$\int_{Q_t}\Big(-\hat{\mathbf{u}}\hat{\mathbf{v}}_t+\nu\hat{\mathbf{u}}_x\hat{\mathbf{v}}_x+l\hat{\mathbf{u}}\hat{\mathbf{v}}-g\rho\int_0^{x_3}\operatorname{div}'\hat{\mathbf{v}}\,dz$$

$$-u_j\hat{\mathbf{u}}\hat{\mathbf{v}}_{x_j}+\int_0^{x_3}\operatorname{div}'\hat{\mathbf{u}}\,dz\,\hat{\mathbf{u}}\hat{\mathbf{v}}_{x_3}\Big)\,dxdt$$

$$+\int_\Omega \hat{\mathbf{u}}\hat{\mathbf{v}}|_{t=t}-\int_\Omega \hat{\mathbf{u}}_0\hat{\mathbf{v}}|_{t=0}=0. \tag{42}$$

Weak solution is a pair of functions $\hat{\mathbf{u}}\in\mathbf{V}_2$, $\rho\in R$ satisfying for any $\hat{\mathbf{v}}\in\mathbf{V}_2$, $r\in R$ and arbitrary $t\in[0,T]$ relations (40), (42).

Using obtained a priori estimates it is not difficult to prove uniqueness of a solution.

As for existence, we use the theorem on existence "in small" and obtained a priori estimates. It gives existence "in the large".

Thus, the following statement is proven

Theorem. *Let* $\hat{\mathbf{u}}_0 \in \mathbf{W}_3^2(\Omega)$, $\rho_0 \in W_2^2(\Omega)$ *satisfy boundary conditions (1") and* $\int_0^1 \mathrm{div}'\hat{\mathbf{u}}_0 dz = 0$. *Then for any* ν, μ, $\nu_v > 0, \mu_v$ *(*μ_v *satisfies the properties discussed above) and arbitrary* $T > 0$ *problem (1) has in* Q_T *a unique weak solution such that* $\hat{\mathbf{u}}^2$, $\hat{\mathbf{u}}_z^2$, $\hat{\mathbf{u}}_x$, $\hat{\mathbf{u}}_{zx}^2$, $\hat{\mathbf{u}}_t$, $\hat{\mathbf{u}}_{tx}$ $\mathbf{L}_2(Q_T)$, *and* ρ^2, ρ_x, ρ_{zx}, $\rho_{tx} \in L_2(Q_T)$. *For* $\hat{\mathbf{u}}$, ρ *estimate (36) holds and the norm* $\|\hat{\mathbf{u}}_x\|$ *is continuous in* t.

Sometimes, one chooses μ_v as

$$\mu_v = \nu_{min} \exp\left\{0.5 \ln \frac{\nu_{max}}{\nu_{min}} \left[1 - th(\gamma \rho_z)\right]\right\}$$

Direct calculations give us the following sufficient restriction

$$\frac{\nu_{max}}{\nu_{min}} \leq 86$$

for the inequality

$$(\mu_v \rho_{zz}, \rho_{zz}) + (\mu_v' \rho_{zz}, \rho_{zz}\rho_z) > 0$$

to be valid. So, the theorem is fulfilled for this case as well.

REFERENCES

1. Lions J.L., Temam R. and Wang S. On the equations of the large-scale ocean. — Nonlinearity, 5, 1992, 1007–1053.

2. Lions J.L., Temam R. and Wang S. New formulations of the primitive equations of the atmosphere and applications. — Nonlinearity, 5, 1992, 237–288.

3. Lewandowski R. Analyse Mathématique et Océnographie. Masson, Paris, 1997.

4. Hu Ch., Temam R. and Ziane M. The primitive equations on the large scale ocean under the small depth hypothesis. — Discrete and continuous dynamical systems, v. 9, Number 1, January 2003.

5. Cao Ch. and Titi E.S. Global Well–posedness of the Three–dimensional Viscous Primitive Equations of Large Scale Ocean and Atmosphere Dynamics. — arXiv:math.AP/0503028, v2, 16 Nov 2005.
6. Temam R. and Ziane M. Some mathematical problems in geophysical fluid dynamics. *Handbook of Mathematical Fluid Dynamics*, vol. **3**, S. Frielander and D. Serr Eds, Elsevier, 2004, 535-658.

7. Kobelkov G.M. Existence of a solution ”in the large” for the 3D large-scale ocean dynamics equations. C. R. Acad. Sci. Paris, Ser. I 343(2006), 283-286.
8. Kobelkov G.M. Existence of a solution ”in the large” for ocean dynamics equations. J. math. fluid mech., 9(2007), 4, 588-610
9. Kobelkov G.M. and Zalesny V.B. Existence and Uniqueness of a Solution to the Primitive Equations with Stratification ”in the large”. Russian J. Numerical Anal. Math. Model., 2008, 1, 39-61.