Uniqueness Through Noise

— Encounters between Analysis and Probability

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Aims of this Lecture

- See similarities and differences between ordinary and stochastic differential equations.
- Glimpse at the connection between stochastic processses and partial differential equations.
- See this connection 'in action' in a specific situation.

1. Introduction

Solving ODE

$$\begin{cases} X'(t) & = f(X(t)) \\ u(0) & = x_0 \end{cases}$$

Search fixed points of

$$X(t) = x_0 + \int_0^t f(X(s))\,ds.$$

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Two cases:

- f Lipschitz $\rightsquigarrow$ use Banach's fixed point theorem.
  - Existence and uniqueness of solutions.
- f merely continuous (or even only measurable) $\rightsquigarrow$ use Compactness (Peano's Theorem)
  - Existence but not necessarily uniqueness of solutions.

Example

Use $f(x) = 2\sqrt{x}$. Then for $x_0 = 0$ both $X_1 \equiv 0$ and $X_2(t) := t^2$ solve

$$\begin{cases} X'(t) &= f(X(t)) \\ X(0) &= 0 \end{cases}$$

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$$\begin{cases} X'(t) & = f(X(t)) \\ X(0) & = 0 \end{cases}$$

The situation is worse if f is merely measurable, e.g. $f(x) = \mathbb{1}_{\mathbb{R}\setminus\mathbb{Q}}(x)$. We want to solve

$$X(t) = x_0 + \int_0^t \mathbb{1}_{\mathbb{R}\setminus\mathbb{Q}}(X(s))\, ds.$$

If $x_0 \in \mathbb{Q}$, then $X_1(t) \equiv x_0$ is a solution and $X_2(t) := x_0 + t$ is a solution.

Uniqueness through speed

Let f be bounded and continuous. Suppose, we want to solve

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$$\begin{aligned} Y(t) &= x_0 + \int_0^t f(X(s))\,ds \\ &= x_0 + \int_0^t (\lambda + f(Y(s) + \lambda s))\Phi'(Y(s) + \lambda s)\,ds \\ &= x_0 + \Phi(Y(t) + \lambda t) + \Phi(x_0). \end{aligned}$$

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Thus, if X_1, X_2 are solutions, we have

$$|Y_1(t) - Y_2(t)| = \big|\Phi(Y_1(t) + \lambda t) - \Phi(Y_2(t) + \lambda t)\big| \leq L|Y_1(t) - Y_2(t)|.$$

$\rightsquigarrow$ Uniqueness if λ is large.

How fast can you go?

Recall: Brownian motion $(B_t)_{t\geq 0}$ is a stochastic process with continuous paths such that

1. $B_0 \equiv 0$.
2. $B_{t+s} - B_t$ is independent of $\mathscr{F}_t := \sigma(B_r : r \leq t)$.
3. $B_{t+s} - B_t \sim \mathscr{N}(0, s)$.

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Thus: consider

$$\begin{cases} X'(t) &= f(X(t)) + \frac{d}{dt}B_t \\ X(0) &= x_0. \end{cases}$$

Of course, this does not make sense. However, we can integrate:

$$X(t) = x_0 + \int_0^t f(X(s))\,ds + \int_0^t \frac{d}{ds}B_s\,ds = x_0 + \int_0^t f(X(s))\,ds + B_t$$

Aim of this lecture

Show uniqueness of solutions for the stochastic integral equation

$$X(t) = x_0 + \int_0^t f(X(s))\,ds + B_t,$$

where f is a bounded, measurable function.
Equivalently, show uniqueness of solutions for the stochastic differential equation

$$dX(t) = f(X(t))dt + dB_t$$

2. Stochastic Differential Equations

Solutions of SDE

Definition

A solution of the SDE is a pair (X, B), defined on a stochastic basis $(\Omega, \Sigma, \mathbb{F}, \mathbb{P})$, where B is an $\mathbb{F}$-Brownian motion and X is a continuous, $\mathbb{F}$-adapted process such that for $t \geq 0$

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If f is Lipschitz continuous, then solutions of the equation

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can be easily constructed by applying the Banach fixed point iteration pathwise, i.e. ω by ω.
Note that we can prescribe the stochastic basis and the Brownian motion (we say the solution exists strongly).

The case of measureable f

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- If f is merely continuous (or measurable), one cannot solve the equation pathwise, as this does not necessarily yields adapted (not even necessarily measurable) processes.
- In general: One cannot construct solutions on a given Probability space/with respect to a given Brownian motion. (One says solutions exist only weakly).
- Classical example by Tanaka (with multiplicative noise):

$$dX(t) = \mathrm{sgn}(X(t))dB_t.$$

  - Solution X has to be a Brownian motion, but it has to be different from B.
  - Let X be a Brownian motion. One constructs a Brownian motion B such that X solves with respect to this Brownian motion.
  - Contradicts notion of causality.

Notions of Uniqueness

As solutions may be defined on different probability spaces, one needs different notions of uniqueness.

- Uniqueness in law (or weak uniqueness):
  If (X_1, B_1) and (X_2, B_2) are solutions, then $X_1(t)$ has the same distribution as $X_2(t)$.

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- Pathwise uniqueness (or strong uniquenss):
  If (X_1, B) and (X_2, B) are solutions (on the same space and wrt the same BM), then $X_1 = X_2$ almost surely.

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Theorem (Yamada-Watanabe)

Weak existence and strong uniqueness imply strong existence and weak uniqueness.

Reformulation of the main result

Theorem (Zvonkin)

Strong uniqueness holds for the stochastic differential equation

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Comments:

- This is only a special case of Zvonkin's result.
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- (weak) existence of solutions can be proved in a standard way. We thus have strong existence by the YW result.
- There is a related result due to Davie:
  There exists a set $\Gamma \subset C([0,1])$ of full Wiener measure such that for $\omega \in \Gamma$, there exists only one solution of

$$X(t) = x_0 + \int_0^t f(X(s))\,ds + \omega(t)$$

3. Partial Differential Equations

The heat equation

Note: $x + B_t \sim \mathscr{N}(x,t)$. Given u_0, put

$$u(t,x) := \mathbb{E}u_0(x+B_t) = \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} u_0(y)e^{-\frac{(y-x)^2}{2t}}\,dy.$$

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Then

$$\begin{aligned}\frac{\partial}{\partial t}u(t,x) &= \frac{-1}{\sqrt{2\pi}}\frac{1}{2}t^{-\frac{3}{2}}\int_{\mathbb{R}} u_0(y)e^{-\frac{(y-x)^2}{2t}}\,dy\\ &\quad + \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} \frac{(y-x)^2}{2t^2}u_0(y)e^{-\frac{(y-x)^2}{2t}}\,dy\\ &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} u_0(y)\frac{1}{2}\Big(t^{-2}(y-x)^2 - t^{-1}\Big)e^{-\frac{(y-x)^2}{2t}}\,dy.\end{aligned}$$

The heat equation cont'd

$$u(t,x) := \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} u_0(y) e^{-\frac{(y-x)^2}{2t}}\, dy.$$

For the x-derivative, we find

$$\frac{\partial}{\partial x} u(t,x) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} u_0(y) \frac{(y-x)}{t} e^{-\frac{(y-x)^2}{2t}}\, dy$$

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so that

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} u(t,x) &= \frac{1}{\sqrt{2\pi t}} \Big(\int_{\mathbb{R}} u_0(y) \frac{-1}{t} e^{-\frac{(y-x)^2}{2t}}\, dy \\
&\qquad + \int_{\mathbb{R}} \frac{(y-x)^2}{t^2} u_0(y) e^{-\frac{(y-x)^2}{2t}}\, dy \Big) \\
&= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} u_0(y) \Big(t^{-2}(y-x)^2 - t^{-1} \Big) e^{-\frac{(y-x)^2}{2t}}\, dy \\
&= 2 \frac{\partial}{\partial t} u(t,x).
\end{aligned}$$

The heat equation cont'd

Thus, *u* solves the Heat equation

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x) = \frac{1}{2}\Delta u(t,x).$$

Note: We can thus solve the partial differential equation (Cauchy problem)

$$\begin{cases} u_t(t,x) & = \frac{1}{2}\Delta u(t,x) \\ u(0,x) & = u_0(x) \end{cases}$$

by computing expected values!

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Question: What happens if instead we use the solution of

$$X(t) = x + \int_0^t f(X(s))\, ds + B_t?$$

The associated PDE

The stochastic differential equation

$$dX(t) = f(X(t))dt + dB_t$$

is associated with the differential operator

$$\mathscr{A}\varphi(x) = f(x)\frac{d}{dx}\varphi(x) + \frac{1}{2}\frac{d^2}{dx^2}\varphi(x).$$

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Connection:

- If one can solve the Cauchy Problem, then one has uniqueness in law for the SDE.
- Here: We use more information about the Cauchy problem (higher regularity) to prove pathwise uniqueness for the SDE.

Solving the Cauchy problem

Result: For bounded and measurable f and continuous u_0 not growning to fast, one can 'solve' the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) & = f(x)u_x(t,x) + \frac{1}{2}u_{xx}(t,x) \\ u(0,x) & = u_0(x) \end{cases}$$

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- ‘Solving’ has to be understood in a weak sense.
- The solution has a certain Sobolev regularity.
- By Sobolev embedding, u_x (but not necessarily u_{xx}) is continuous.

4. Stochastic calculus in a nutshell

Itô's Integral

Let B_t be a Brownian motion with respect to the filtration $\mathbb{F}$. An elementary step process is a process of the form

$$\Phi(t,\omega) := \sum_{k=1}^{n} \eta_k(\omega)\mathbb{1}_{[t_{k-1},t_k)}(t)$$

where $0 \leq t_0 < t_1 < \cdots t_k \leq T$ and η_k is $\mathscr{F}_{t_{k-1}}$-measurable.

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$$\int_0^T \Phi(t)\, dB_t := \sum_{k=1}^{n} \eta_k(B_{t_k} - B_{t_{k-1}}).$$

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$$\int_0^T \Phi(t)\, dB_t := \sum_{k=1}^{n} \eta_k(B_{t_k} - B_{t_{k-1}}).$$

One can prove the Itô isometry

$$\mathbb{E}\Big|\int_0^T \Phi(t)\, dB_t\Big|^2 = \mathbb{E}\int_0^T |\Phi(t)|^2\, dt.$$

Thus, we can extend the integral to the closure of elementary step processes in $L^2(\Omega \times (0,T))$.

Itô's Formula

Let X satisfy the SDE

$$dX(t) = f(X(t))dt + dB_t$$

and let $u \in C^{1,2}$. Then we can write an SDE for $Y(t) = u(t, X(t))$, namely

$$du(t, X(t)) = \Big[u_t(t, X(t)) + f(X(t))u_x(t, X(t)) + \frac{1}{2}u_{xx}(t, X(t))\Big]dt + u_x(X(t))dB_t$$

which has to be understood in integral form, i.e.

$$u(t, X(t)) = u(0, X(0)) + \int_0^t \Big[u_t(s, X(s)) + f(X(s))u_x(s, X(s)) + \frac{1}{2}u_{xx}(s, X(s))\Big]\, ds + \int_0^t u_x(s, X(s))\, dB_s$$

Some remarks on Itô's formula

Case with no stochastic term:
If $X'(t) = f(X(t))$, then

$$\begin{aligned} \frac{d}{dt}u(t, X(t)) &= u_t(t, X(t)) + u_x(t, X(t))X'(t) \\ &= u_t(t, X(t)) + u_x(t, X(t))f(X(t)) \end{aligned}$$

and Itô's formula reduces to the fundamental theorem of calculus.

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and Itô's formula reduces to the fundamental theorem of calculus.
Naive approach:

$$\begin{aligned}du(t,X(t)) &= u_t(t,X(t))dt + u_x(t,X(t))dX(t)\\ &= u_t(t,X(t)dt + u_x(t,X(t))f(X(t))dt + u_x(t,X(t))dB_t.\end{aligned}$$

Difference to Itô's formula: $\frac{1}{2}u_{xx}(s,X(s))dt$ Itô correction (due to use of Itô integral).

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5. Showdown

Proof of the main result

Theorem

Let f be a bounded, measurable function. Then pathwise uniqueness holds for the solutions of the equation

$$dX(t) = f(X(t))dt + dB_t$$

Proof:

Let X_1, X_2 be two solutions on the same probability space and with respect to the same Brownian motion.

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Let X_1, X_2 be two solutions on the same probability space and with respect to the same Brownian motion.

Let u be the solution of

$$\begin{cases} \frac{\partial}{\partial t} u & = -\mathscr{A} u \\ u(T) & = id \end{cases}.$$

By Itô's formula:

$$\begin{aligned} u(t, X_j(t)) &= \left[u_t(t, X_j(t)) + \mathscr{A} u(t, X(t))\right] dt + u_x(t, X_j(t)) dB_t \\ &= u_x(t, X_j(t)) dB_t. \end{aligned}$$

Proof cont'd

Thus

$$u(t, X_1(t)) - u(t, X_2(t)) = \int_0^t u_x(s, X_1(s)) - u_x(s, X_2(s))\, dB_s.$$

By the Itô isometry

$$\mathbb{E}\big[u(t, X_1(t)) - u(t, X_2(t))\big]^2 = \mathbb{E}\int_0^t \big[u_x(s, X_1(s)) - u_x(s, X_2(s))\big]^2\, ds.$$

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Note that

$$\begin{aligned} &u_x(s, X_1(s)) - u_x(s, X_2(s)) \\ &= \int_0^1 u_{xx}(s, \sigma X_1(s) + (1-\sigma)X_2(s)\, d\sigma \cdot (X_1(s) - X_2(s)). \end{aligned}$$

By regularity of the solution, $\int_0^1 u_{xx}$ is bounded.

Proof cont'd

Thus,

$$\mathbb{E}\big[u(t, X_1(t)) - u(t, X_2(t))\big]^2 \leq C \int_0^t \mathbb{E}(X_1(s) - X_2(s))^2 \, ds.$$

Proof cont'd

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$$\mathbb{E}\big[u(t, X_1(t)) - u(t, X_2(t))\big]^2 \leq C \int_0^t \mathbb{E}(X_1(s) - X_2(s))^2\, ds.$$

Studying the regularity of u, it follows that $|u(t,x) - u(t,y)| \geq \alpha|x - y|$ for some $\alpha > 0$. Consequently,

$$\begin{aligned}\mathbb{E}\big[X_1(t) - X_2(t))\big]^2 &\leq \alpha^{-2}\mathbb{E}(u(t, X_1(t)) - u(t, X_2(t))^2 \\ &\leq \alpha^{-2} C \int_0^t \mathbb{E}(X_1(s) - X_2(s))^2\, ds.\end{aligned}$$

Proof cont'd

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By Gronwall's Lemma $\mathbb{E}(X_1(t) - X_2(t))^2 \equiv 0$. Thus $X_1(t) = X_2(t)$ almost surely for every t.

By continuity of the paths, $X_1 \equiv X_2$ almost surely. QED.

References

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