On the Coefficients of Expansions in Bases of Smooth Functions

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The Definition I

Definition. A function $f(\mathbf{x})$ given on the cube $[0,\ 1]^d$ belongs to the class *Lip* α (α-Lipschitz) for $0 < \alpha \leq 1$ if there exists a constant M such that, for all $\mathbf{x}$ and $\mathbf{y}$

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq M\rho^{\alpha}(\mathbf{x},\ \mathbf{y})$$

where $\rho(\mathbf{x},\ \mathbf{y})$ is the Euclidean distance between the points.
The function $f(\mathbf{x})$ belongs to the class *Lip* α for $\alpha = p + \gamma$, where $p \in \mathbb{N}$ and $0 < \gamma \leq 1$, if all of its derivatives of order $0 \leq |\mathbf{s}| < p$, $\mathbf{s} = (s_1, ..., s_d)$,

$$D^{\mathbf{s}} f = \frac{d^{|\mathbf{s}|} f(\mathbf{x})}{dx_1^{s_1} ... dx_d^{s_d}}$$

exist and are continuous; all the higher derivatives $D^{\mathbf{s}} f$, where $|\mathbf{s}| = p$, belong to the class *Lip* γ.

Function of a single variable I

The following theorem on functions of a single variable is well known.

Theorem 1 (Bernstein) If $f(x) \in Lip\ \alpha$, $\alpha > \frac{1}{2}$, is a periodic function, then the series of its Fourier coefficients in the trigonometric system is absolutely convergent.

There exists a periodic function $f(x) \in Lip\ \frac{1}{2}$ such that the series of the Fourier coefficients in its trigonometric system is not absolutely convergent.

Function of a single variable II

In 1964, Mityagin obtained a generalization of the second part of Bernstein's theorem to the case of an arbitrary complete orthonormal system. The following statement holds.

Theorem 2 (Mityagin) Suppose that $\Psi = \{\psi_n(x)\}_{n=1}^{\infty}$ is an arbitrary complete orthonormal system on $[0,\ 1]$. There exists a function $f(x) \in Lip\ \frac{1}{2}$ for which the series of the Fourier coefficients in the system Ψ is not absolutely convergent.

Function of a single variable III

In 1977, Kashin proved the validity of a similar result for an arbitrary normalized basis in $L_p([0, 1])$, $1 < p < \infty$.

Theorem 3 (Kashin) Suppose that $\Psi = \{\psi_n(x)\}_{n=1}^{\infty}$ is a basis in $L_p([0, 1])$, $1 < p < \infty$, and $||\psi_n(x)||_p = 1$, $n = 1, 2, \ldots$. There exists a function $f(x) \in Lip\ \alpha$, $\alpha = \min\left\{\frac{1}{2}, \frac{1}{q}\right\}$, $\frac{1}{q} + \frac{1}{p} = 1$, such that the series of the modules of the coefficients of the expansion of the function $f(x)$ in the basis Ψ is divergent, i.e., $f(x) = \sum_{n=1}^{\infty} a_n \psi_n(x)$, then $\sum_{n=1}^{\infty} |a_n| = \infty$.

The accuracy of this theorem is confirmed by the examples of the trigonometric system (for $p \geq 2$) and of the normalized Haar system in $L_p([0, 1])$ (for $1 < p \leq 2$).

Functions of several variables I

Analogs of Bernstein's theorem are also known for functions of several variables. The following results are valid for the multiple trigonometric system.

Theorem 4 (Bochner) For any periodic function $f(\mathbf{x}) \in Lip\ \left(\frac{d}{2} + \varepsilon\right)$, $\varepsilon > 0$, given on $[0, 1]^d$, the series of the Fourier coefficients in the trigonometric system is absolutely convergent.

Theorem 5 (Wainger) There exists a periodic function $f(\mathbf{x}) \in Lip\ \frac{d}{2}$ given on $[0,\ 1]^d$ for which the series of the Fourier coefficients in the trigonometric system is not absolutely convergent.

Functions of several variables II

The analog of Theorem 5 for the case of an arbitrary complete orthonormal system of functions of several variables is given. Let us present a particular case of the result of Mityagin.

Theorem 6 (Mityagin) Suppose that $\Psi = \{\psi_n(\mathbf{x})\}_{n=1}^{\infty}$ is an arbitrary complete orthonormal system on $[0,\ 1]^d$. There exists a function $f(\mathbf{x}) \in Lip\ \frac{d}{2}$ for which the series of the Fourier coefficients in the system Ψ is not absolutely convergent.

Functions of several variables III

The following theorem is a similar result for an arbitrary normalized basis in the space $L_p([0,\ 1]^d)$, $1 < p < \infty$.

Theorem 7 (A.M.) Suppose that $\Psi = \{\psi_n(\mathbf{x})\}_{n=1}^{\infty}$ is an arbitrary normalized basis in the space $L_p([0,\ 1]^d)$, $1 < p < \infty$. Suppose that $\alpha = \min\left\{\frac{1}{2}, \frac{1}{q}\right\}$, $\frac{1}{q} + \frac{1}{p} = 1$.

1. If $d\alpha \notin \mathbb{Z}$, then there exists a function $f(\mathbf{x}) \in Lip\ d\alpha$ for which the series of modules of the coefficients of the expansion in the basis Ψ is divergent.
2. If $d\alpha \in \mathbb{Z}$, then there exists a function $f(\mathbf{x}) \in Lip\ (d\alpha - \varepsilon)$ for which the series of modules of the coefficients of the expansion in the basis Ψ is divergent.

Functions of several variables IV

Some remarks about proof of Th.7

We inductively define the following systems of functions:

$$\varphi^i_{0,k}(x), i = 1, 2, ..., 2^k, k = 1, 2, ...$$

- are the functions of the Faber–Schauder system;

$$\varphi^i_{l,k}(x) = 2^{k+2} \int_x^{x+\frac{1}{2^{k+1}}} \varphi^{2i}_{l-1,k+1}(t) dt,$$

where $i = 1, 2, ..., 2^k,\ k = 1, 2, ...,\ l = 1, 2, ..., m.$

Functions of several variables V

For the systems of the functions $\{\varphi^i_{l,k}(x)\}$, $i = 1, ..., 2^k$, $k = 1, 2, ...$ the following statement is valid for any $l = 0, 1, ...$ (this result was obtained by Chiselskii for the Faber–Schauder system).

Lemma. If the coefficients of the series

$$f(x) = \sum_{k=1}^{\infty} \sum_{i=1}^{2^k} A_{k,i} \varphi^i_{l,k}(x)$$

satisfy the relation $|A_{k,i}| \leq C \frac{1}{2^{k\gamma}}$, $0 < \gamma < 1$, then $f(x) \in Lip\ \gamma$.

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Thank you for your attention!