On an Inequality of Sidon Type for Trigonometric Polynomials

A. O. Radomskii
Moscow State University

Abstract–We establish a lower bound for the uniform norm of the trigonometric polynomial of special form via the sum of the L^1-norm of its summands. This result generalizes a theorem due to Kashin and Temlyakov, which, in turn, generalizes the classical Sidon inequality.

We set, as usual, by $\|f\|_p$ the norm in $L^p(0,2\pi)$, $1 \le p \le \infty$:

$$\|f\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p\,dx\right)^{\frac{1}{p}}, \ 1 \le p < \infty,$$
$$\|f\|_\infty = \operatorname*{ess\,sup}_{[0,2\pi]} |f(x)|.$$

By the classical Sidon theorem (see [6]; pp. 393-395) if the sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$, is lacunary with lacunarity index λ (i.e., $n_{k+1}/n_k \ge \lambda > 1$ for any k), then there exists a positive constant $c(\lambda)$ depending only on the lacunarity index such that, for any trigonometric polynomial of the form

$$T(x) = \sum_{k=1}^{N} a_k \cos n_k x, \qquad a_k \in \mathbb{R},$$

the following inequality holds:

$$\|T\|_\infty \ge c(\lambda)\sum_{k=1}^{N} |a_k|.$$

Kashin and Temlyakov obtained a more general result in the special case. Namely, they proved (see [1] and also [2])

Theorem A (B.S. Kashin, V.N. Temlyakov). *For any trigonometric polynomial of the form*

$$f(x) = \sum_{k=l+1}^{2l} p_k(x)\cos 4^k x,$$

where the p_k *are real trigonometric polynomials of* $\deg(p_k) \le 2^l$, *where* $k = l+1, \dots, 2l$ *and* $l = 1, 2, \dots,$ *the following inequality holds*

$$\|f\|_\infty \ge c \sum_{k=l+1}^{2l} \|p_k\|_1,$$

where $c > 0$ *is an absolute constant.*

Kashin and Temlyakov also introduced QC-norm (see [1] and [2]) and proved that for this norm the previous result is true in a more general case:

for a function $f \in L^1(0, 2\pi)$ with Fourier series $f \sim \sum_{s=0}^{\infty} \delta_s(f, x)$, where

$$\delta_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx, \qquad \delta_s = \sum_{2^{s-1} \le |k| < 2^s} \hat{f}(k) e^{ikx},\ s = 1, 2, \dots,$$

QC-norm of f is defined as

$$\|f\|_{QC} \equiv \int_0^1 \left\| \sum_{s=0}^{\infty} r_s(\omega) \delta_s(f, \cdot) \right\|_{L^\infty} d\omega,$$

where $\{r_k(\omega)\}_{k=0}^{\infty}$ is the system of Rademacher.

Theorem B (B. S. Kashin, V. N. Temlyakov). *For any real function* $f \in L^1(0, 2\pi)$ *the following inequality holds*

$$\|f\|_{QC} \ge \frac{1}{16} \sum_{s=0}^{\infty} \|\delta_s(f)\|_1.$$

A similar result to the theorem A is proved by the author in the case of an arbitrary lacunarity index and under weaker constraints on the degrees of the polynomials p_k (see [3] and [5]).

Theorem 1. *Suppose that a sequence of natural numbers* $\{n_k\}_{k=1}^{\infty}$ *satisfies the condition* $n_{k+1}/n_k \ge \lambda > 1$, $k = 1, 2, \dots$. *There exist the constants* $c = c(\lambda) > 0$ *and* $\gamma = \gamma(\lambda) \in (0, 1)$ *depending only on* λ *such that, for any trigonometric* *polynomial of the form*

$$f(x) = \sum_{k=l}^{N} p_k(x) \cos n_k x,$$

where p_k *are real trigonometric polynomials of* $\deg(p_k) \leq [\gamma n_l]$, $k = l, \ldots, N$, $N \geq l$, $l = 1, 2, \ldots,$ *the following inequality holds*

$$\|f\|_\infty \geq c \sum_{k=l}^{N} \|p_k\|_1.$$

In connection with research of QC-norm Grigor'ev proved the following result (see [4]).

Theorem (Grigor'ev). *There exists the sequence of trigonometric polynomials* $\{\sigma_k(x)\}_{k=1}^{\infty}$ *such that*

$$\sigma_k(x) = \sum_{2^k \leq |j| < 2^{k+1}} c_j e^{ijx}, \ \|\sigma_k\|_1 \geq \frac{\pi}{4}, \ \|\sigma_k\|_\infty \leq 6, \ k = 1, 2, \ldots,$$

and

$$\Big\|\sum_{k=1}^{n} \sigma_k\Big\|_\infty \leq A\sqrt{n}, \ n = 1, 2, \ldots,$$

where $A > 0$ *is an absolute constant.*

Using some ideas of Grigor‘ev‘s work we have proved (see [5]) the following result which shows that in theorem 1 (in the case $n_k = 2^k$) the condition $\deg(p_k) \leq [\gamma 2^l]$ can‘t be substituted for $\deg(p_k) \leq [2^{k-k^\varepsilon}]$ neither any $\varepsilon \in (0,1)$.

Theorem 2. *Suppose that* ε, $\widetilde{\varepsilon}$ *are real numbers such that* $\frac{1}{2} < \varepsilon < \widetilde{\varepsilon} < 1$. *For any* $W \in \mathbb{N}$ *there exist the real trigonometric polynomials* $p_k(x)$, $k = 1, \ldots, W$, *such that* $\deg p_k \leq \left[2^{k-k^\varepsilon}\right]$, $\|p_k\|_1 \geq \frac{1}{3}$, $\|p_k\|_\infty \leq 70$, $k = 1, \ldots, W$, *and*

$$\max_{1 \leq n \leq W} \left\|\sum_{k=1}^{n} p_k(x) \cos 2^k x\right\|_\infty \leq C W^{\widetilde{\varepsilon}},$$

where $C = C(\varepsilon, \widetilde{\varepsilon}) > 0$ *is a constant depending only on* ε *and* $\widetilde{\varepsilon}$.

REFERENCES

[1] B. S. Kashin, V. N. Temlyakov, On a norm and approximation characteristics of classes of functions of several variables, in *Metric Theory of Functions and Related Problems in Analysis*, Collection of articles dedicated to P. L. Ul'yanov on his seventieth birthday (AFTs, Moscow, 1999), pp. 69–99 [in Russian].

[2] B. S. Kashin, V. N. Temlyakov, "On a certain norm and related applications", Math. Notes, Vol. 64, No. 4 (1998), pp. 551–554.

[3] A. O. Radomskii, "On an Inequality of Sidon Type for Trigonometric Polynomials", Math. Notes, 2011, Vol. 89, No. 4, pp. 555-561.

[4] P. G. Grigor'ev, "On a sequence of trigonometric polynomials", Math. Notes, 1997, Vol. 61, No. 6, pp. 780-783.

[5] A. O. Radomskii, "On the possibility of an extension of inequalities of Sidon type", Mat. Zametki, (in press).

[6] A. Zygmund Trigonometric Series. V. 1. M.: Mir, 1965 [in Russian].